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Funktionalanalysis
WiSe 2020–21

Problem Set 4

Due: Thursday, 3.12.2020 (22pts total)

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Convention: *Unless otherwise stated, you can assume in every problem that (X, μ) is an arbitrary measure space and functions in $L^p(X) := L^p(X, \mu)$ take values in a fixed finite-dimensional inner product space $(V, \langle\ ,\ \rangle)$ over a field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. Whenever X is a subset of $\mathbb{R}^n$, you can also assume by default that μ is the Lebesgue measure m.*

Problem 1 (∗)
Assume $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove:

(a) For any closed subspace $E \subset L^p(X)$ with $E \neq L^p(X)$, there exists a function $g \in L^q(X) \backslash \{0\}$ that satisfies $\int_X \langle g, f \rangle \, d\mu = 0$ for every $f \in E$.
Hint: Since $L^p(X)$ is uniformly convex, there exists a closest point in E to any given point in $L^p(X) \backslash E$. [6pts]

(b) A linear subspace $E \subset L^p(X)$ is dense if and only if every bounded linear functional $\Lambda : L^p(X) \to \mathbb{K}$ that vanishes on E is trivial. [3pts]

Comment: The result of this problem is often used in applications and cited as a consequence of the Hahn-Banach theorem, which implies a similar result for subspaces of arbitrary Banach spaces. However, the uniform convexity of $L^p(X)$ for $1 < p < \infty$ makes the use of the Hahn-Banach theorem (which relies on the axiom of choice) unnecessary in this setting. You should not use it in your solution either, since we have not proved it yet.

Problem 2
Show that if $f \in L^\infty(X)$ satisfies $|f| < \|f\|_{L^\infty}$ almost everywhere, then

$$\left| \int_X \langle g, f \rangle \, d\mu \right| < \|g\|_{L^1} \cdot \|f\|_{L^\infty} \quad \text{for every} \quad g \in L^1(X) \backslash \{0\},$$

i.e. there is *strict* inequality. Give an example $f \in L^\infty([0, 1])$ satisfying this condition.
Hint: What can you say about $\int_X (c - |f|)|g| \, d\mu$ if $|f| < c$ almost everywhere?
Comment: The Hahn-Banach theorem implies that for every nontrivial element x in a Banach space E, there exists a bounded linear functional $\Lambda \in E^*$ with $\|\Lambda\| = 1$ and $\Lambda(x) = \|x\|$. For $E = L^\infty(X)$, it follows that this $\Lambda \in E^*$ cannot be represented as $\Lambda_g = \int_X \langle g, \cdot \rangle \, d\mu$ for any $g \in L^1(X)$. This is one way of seeing that the Riesz representation theorem is false for $p = \infty$.

Problem 3

(a) Show that if (M, d) is a metric space containing an uncountable subset $S \subset M$ such that every pair of distinct points $x, y \in S$ satisfies $d(x, y) \geqslant \epsilon$ for some fixed $\epsilon > 0$, then M is not separable.

(b) Suppose (X, μ) contains infinitely many disjoint subsets with positive measure. Show that $L^\infty(X)$ contains an uncountable subset $S \subset L^\infty(X)$, consisting of functions that take only the values 0 and 1, such that $\|f - g\|_{L^\infty} = 1$ for any two distinct $f, g \in S$. Conclude that $L^\infty(X)$ is not separable.
Hint: If you've forgotten or never seen the proof via Cantor's diagonal argument that $\mathbb{R}$ *is uncountable, looking it up may help.*

(c) Let $\mathscr{L}(\mathcal{H})$ denote the Banach space of bounded linear operators $\mathcal{H} \to \mathcal{H}$ on a separable Hilbert space $\mathcal{H}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Show that any orthonormal basis $\{e_n\}_{n=1}^\infty$ of $\mathcal{H}$ gives rise to a continuous linear inclusion

$$\Psi : \ell^\infty \hookrightarrow \mathscr{L}(\mathcal{H}),$$

where ℓ^∞ denotes the Banach space of bounded sequences $\{\lambda_n \in \mathbb{K}\}_{n=1}^\infty$ with norm $\|\{\lambda_n\}\|_{\ell^\infty} := \sup_{n \in \mathbb{N}} |\lambda_n|$, and $\Psi(\{\lambda_n\}) \in \mathscr{L}(\mathcal{H})$ is uniquely determined by the condition $\Psi(\{\lambda_n\})e_j := \lambda_j e_j$ for all $j \in \mathbb{N}$.

Comment: It is not hard to show that every subset of a separable metric space is also separable. Since $\ell^\infty = L^\infty(\mathbb{N}, \nu)$ for the counting measure ν, parts (b) and (c) thus imply that $\mathscr{L}(\mathcal{H})$ is not separable.

Problem 4 $(*)$
This problem deals with *weak* convergence $x_n \rightharpoonup x$. Assume $\mathcal{H}$ is a separable Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with orthonormal basis $\{e_n\}_{n=1}^\infty$, and consider a sequence of the form $x_n := \lambda_n e_n \in \mathcal{H}$ for some $\lambda_n \in \mathbb{K}$. Prove:

(a) $x_n \rightharpoonup 0$ whenever the sequence λ_n is bounded. [3pts]

(b) If the sequence λ_n is unbounded, then x_n is not weakly convergent. [5pts]
Hint: Show that $\lim_{n\to\infty}\langle e_j, x_n\rangle = 0$ for every $j \in \mathbb{N}$ and conclude that if $x_n \rightharpoonup x$ then $x = 0$. Then associate to any subsequence with $|\lambda_{n_j}| \geqslant j$ for $j = 1, 2, 3, \ldots$ an element of the form $v = \sum_{j=1}^\infty a_j e_{n_j} \in \mathcal{H}$ such that $\langle v, x_{n_j}\rangle \not\to 0$ as $j \to \infty$.
Remark: We will later use a general result called the "uniform boundedness principle" to show that weakly convergent sequences must always have bounded norms. But you should not use that result here, since we have not proved it.

(c) If $|\lambda_n| \leqslant \sqrt{n}$ for all $n \in \mathbb{N}$, then every weakly open neighborhood of $0 \in \mathcal{H}$ contains infinitely many elements of the sequence x_n. [5pts]

Comment: If the weak topology on $\mathcal{H}$ were metrizable, then one could deduce from part (c) that a subsequence of $\sqrt{n}e_n$ converges weakly to 0, contradicting part (b). It follows therefore that the weak topology on an infinite-dimensional Hilbert space is not metrizable.

Problem 5
Find a sequence $f_n \in L^p(\mathbb{R})$ for $1 < p < \infty$ that converges weakly to 0 but satisfies $\|f_n\|_{L^p} = 1$ for all n, and deduce that f_n has no L^p-convergent subsequence.

Problem session 5

Problem 1 ($*$)
Assume $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove:

(a) For any closed subspace $E \subset L^p(X)$ with $E \neq L^p(X)$, there exists a function $g \in L^q(X) \backslash \{0\}$ that satisfies $\int_X \langle g, f \rangle \, d\mu = 0$ for every $f \in E$.
Hint: Since $L^p(X)$ is uniformly convex, there exists a closest point in E to any given point in $L^p(X) \backslash E$. [6pts]

(b) A linear subspace $E \subset L^p(X)$ is dense if and only if every bounded linear functional $\Lambda : L^p(X) \to \mathbb{K}$ that vanishes on E is trivial. [3pts]

a) Go through the proof of Riesz rep. thm for $L^p(x)$.

$E \subset L^p(x)$ closed. Choose $h \in L^p(x) \setminus E$

$\because$ $L^p(x)$ is uniformly convex $\Rightarrow$ $\exists\, f_0 \in E$ minimizing the distance to h.

Suppose $f \in E$ arbitrary

$$\frac{d}{dt} \| h - (f_0 - tf) \|_p^p \Big|_{t=0} = 0$$

$$\Rightarrow 0 = p \int_X |h - f_0|^{p-2} \langle h - f_0, f \rangle \, d\mu$$

$$\Rightarrow \int \underbrace{|h - f_0|^{p-2} \langle h - f_0}_{\in L^q(x) \setminus \{0\}}, f \rangle \, d\mu = 0 \quad \forall\, f \in E$$

$\neq 0$ because $h \in L^p(x) \setminus E$ and $f_0 \in E$

$$\frac{1}{p} + \frac{1}{q} = 1 \implies q = \frac{p}{p-1}$$

$$\left| |h-f_0|^{p-2} \, h-f_0 \right| = |h-f_0|^{p-1}$$

$$\because \quad h-f_0 \in L^p(X) \implies \int_X |h-f_0|^p \, d\mu < \infty$$

$$\implies |h-f_0|^{p-2}(h-f_0) \in L^q(X) \setminus \{0\}$$

$\boxed{//}$

Suppose Hahn-Banach

X, E, $E \neq X$, $x_0 \in X \setminus E$

$\mathrm{span}(E, x_0)$, $y = e + \alpha x_0$

(b) Suppose E is dense.

$\Lambda : L^p(X) \longrightarrow \mathbb{K}$ s.t. $\Lambda(E) \equiv 0$

Let $f \in L^p(X) \implies \underset{\substack{\cap \\ E}}{f_n} \xrightarrow[n \to \infty]{} f$

$\Lambda(f_n) \longrightarrow \Lambda(f)$ but $\Lambda(f_n) = 0$

$\implies \Lambda \equiv 0$ on $L^p(X)$.

conversely. suppose E is not dense.

$\Rightarrow \quad \overline{E} \neq L^p(X)$

contradicts problem (a) as we can find non-zero $g \in L^q(X)$ s.t. $g(\overline{E}) = 0$.

$\therefore$ E must be dense. $\blacksquare$

Problem 2
Show that if $f \in L^\infty(X)$ satisfies $|f| < \|f\|_{L^\infty}$ almost everywhere, then

$$\left|\int_X \langle g, f\rangle \, d\mu\right| < \|g\|_{L^1} \cdot \|f\|_{L^\infty} \quad \text{for every} \quad g \in L^1(X)\backslash\{0\},$$

i.e. there is *strict* inequality. Give an example $f \in L^\infty([0,1])$ satisfying this condition.

$g \in L^1(X) \setminus \{0\}$, $\exists$ some $A \subset X$ w/ $\mu(A) > 0$ s.t. $g|_A \not\equiv 0$

$$\left|\int_A \langle g, f\rangle \, d\mu\right| \leq \int_A |\langle g, f\rangle| \, d\mu = \int_A |g||f| \, d\mu$$

$$\Rightarrow \quad -\left|\int_A \langle g, f\rangle \, d\mu\right| \geq -\int_A |g||f| \, d\mu$$

Adding $\|g\|_{L^1} \|f\|_{L^\infty}$

$$\underbrace{\|g\|_{L^1}\|f\|_{L^\infty} - \left|\int_A \langle g, f\rangle \, d\mu\right|}_{>0} \geq \|g\|_{L^1}\|f\|_{L^\infty} - \underbrace{\int_A |g||f| \, d\mu}_{>0}$$

e.g. $f(x) = x$ on $[0,1]$

$\|f\|_{L^\infty} = \text{ess sup } f = 1$

$|f| = \|f\|_{L^\infty}$ on $\{1\}$ - measure zero.

$\therefore \quad |f| < \|f\|_{L^\infty}$ a.e.

Problem 3

(a) Show that if (M, d) is a metric space containing an uncountable subset $S \subset M$ such that every pair of distinct points $x, y \in S$ satisfies $d(x, y) \geqslant \epsilon$ for some fixed $\epsilon > 0$, then M is not separable.

(b) Suppose (X, μ) contains infinitely many disjoint subsets with positive measure. Show that $L^\infty(X)$ contains an uncountable subset $S \subset L^\infty(X)$, consisting of functions that take only the values 0 and 1, such that $\|f - g\|_{L^\infty} = 1$ for any two distinct $f, g \in S$. Conclude that $L^\infty(X)$ is not separable.
Hint: If you've forgotten or never seen the proof via Cantor's diagonal argument that $\mathbb{R}$ is uncountable, looking it up may help.

(c) Let $\mathscr{L}(\mathcal{H})$ denote the Banach space of bounded linear operators $\mathcal{H} \to \mathcal{H}$ on a separable Hilbert space $\mathcal{H}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Show that any orthonormal basis $\{e_n\}_{n=1}^\infty$ of $\mathcal{H}$ gives rise to a continuous linear inclusion

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where ℓ^∞ denotes the Banach space of bounded sequences $\{\lambda_n \in \mathbb{K}\}_{n=1}^\infty$ with norm $\|\{\lambda_n\}\|_{\ell^\infty} := \sup_{n\in\mathbb{N}} |\lambda_n|$, and $\Psi(\{\lambda_n\}) \in \mathscr{L}(\mathcal{H})$ is uniquely determined by the condition $\Psi(\{\lambda_n\})e_j := \lambda_j e_j$ for all $j \in \mathbb{N}$.

a) (M, d) metric space.

Suppose M is seperable $\Rightarrow \exists$ countable dense set $\{m_n\}_{n\in\mathbb{N}}$

$\forall$ $s \in S$, $B_{\frac{\epsilon}{2}}(x_s) \cap \{m_n\}_{n\in\mathbb{N}} \neq \phi$

so choose $n(s) \in \mathbb{N}$ s.t. $m_{n(s)} \in B_{\frac{\epsilon}{2}}(x_s)$.

$s \longmapsto n(s)$ is injective.

if $n(s) = n(t)$

$\Rightarrow$ $m_{n(s)} = m_{n(t)} \in B_{\frac{\epsilon}{2}}(x_s) \cap B_{\frac{\epsilon}{2}}(x_t)$

$\Rightarrow$ $x_s = x_t$

$S \hookrightarrow \mathbb{N}$ contradiction as S is uncountable

$\Rightarrow$ M is not separable. $\boxtimes$

b). S = set of sequences containing only 0's and 1's is uncountable.

<u>Cantor's Diagonal Argument</u>

Suppose S is countable.

$\Rightarrow$ $\exists$ bijection b/w S and $\mathbb{N}$.

$$\begin{array}{l}
s_1 = 0\ 1\ 1\ 0\ 1 \ldots \\
s_2 = 0\ 0\ 0\ 1\ 1 \ldots \\
s_3 = 1\ 1\ 1\ 0\ 1 \ldots \\
s_4 = 1\ 1\ 1\ 1\ 1, \ldots \\
s_5 = 0\ 0\ 1\ 0 \ldots \\
\vdots
\end{array}$$

$t = 1100\ldots$

look at s_{ii}-th entry in s_i

$t \notin S$ because if $t = s_j$ for j

then the j-th entry of $t \neq s_{jj}$-th entry of s_j

$\therefore$ S is uncountable.

$f \in L^\infty(X)$

$\{X_i\}_{i \in I}$ disjoint subsets of X w/ positive measure.

χ_{X_i}

c) $\{e_n\}_{n=1}^{\infty}$ o.n. basis of $\mathcal{H}$.

Define $\Psi : l_\infty \longrightarrow \mathcal{L}(\mathcal{H})$ by , $\lambda \in l_\infty$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{(\lambda_n)}$

$\Psi_\lambda : \mathcal{H} \longrightarrow \mathcal{H}$ by

$\Psi_\lambda(e_n) = \lambda_n e_n$ → clearly an inclusion.

Check linear.

$$\|\Psi_\lambda\|^2 = \sup_{\|h\|=1} \{ \|\Psi_\lambda(h)\|^2 \}$$

$$= \sup_{\|h\|=1} \left\{ \left\| \sum_{n=1}^{\infty} \lambda_n \alpha_n e_n \right\|^2 \right\}$$ where $h = \sum_{n=1}^{\infty} \alpha_n e_n$

$$\le \sup_n |\lambda_n| = \|\lambda\|_{l^\infty} < \infty. \Rightarrow \sum |\alpha_n|^2 = 1$$

Ψ_λ is an inclusion of $l^\infty \hookrightarrow \mathcal{B}(\mathcal{H})$.

Problem 4 (*)

This problem deals with *weak* convergence $x_n \rightharpoonup x$. Assume $\mathcal{H}$ is a separable Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and consider a sequence of the form $x_n := \lambda_n e_n \in \mathcal{H}$ for some $\lambda_n \in \mathbb{K}$. Prove:

(a) $x_n \rightharpoonup 0$ whenever the sequence λ_n is bounded. [3pts]

(b) If the sequence λ_n is unbounded, then x_n is not weakly convergent. [5pts]
Hint: Show that $\lim_{n\to\infty}\langle e_j, x_n\rangle = 0$ for every $j \in \mathbb{N}$ and conclude that if $x_n \rightharpoonup x$ then $x = 0$. Then associate to any subsequence with $|\lambda_{n_j}| \geqslant j$ for $j = 1, 2, 3, \ldots$ an element of the form $v = \sum_{j=1}^{\infty} a_j e_{n_j} \in \mathcal{H}$ such that $\langle v, x_{n_j}\rangle \nrightarrow 0$ as $j \to \infty$.
Remark: We will later use a general result called the "uniform boundedness principle" to show that weakly convergent sequences must always have bounded norms. But you should not use that result here, since we have not proved it.

(c) If $|\lambda_n| \leqslant \sqrt{n}$ for all $n \in \mathbb{N}$, then every weakly open neighborhood of $0 \in \mathcal{H}$ contains infinitely many elements of the sequence x_n. [5pts]

(a) $x_n = \lambda_n e_n$, $\lambda_n \in \mathbb{K}$.

Let $x \in \mathcal{H}$. be an arbitrary.

Then

$$\sum_{n=1}^{\infty} |\langle x, \underset{\lambda_n e_n}{x_n}\rangle|^2 \leq M\|x\|^2$$

where M is the bound on $\{\lambda_n\}$.

$\Rightarrow$ $|\langle e_n, x\rangle|^2 \to 0$ $\Rightarrow$ $\langle e_n, x\rangle \to 0$

$\Rightarrow$ $x = 0$

$x_n \longrightarrow 0$ if $\{\lambda_n\}$. is bounded.

b) $|\lambda_{n_j}| \geq j \quad \forall j = 1, 2, \ldots$

$$v = \sum_{j=1}^{\infty} \frac{1}{j^{1+\epsilon}} e_{n_j} \in \mathcal{H}.$$

v is absolutely convergent

$$|\langle v, x_{n_j} \rangle| = \frac{1}{j^{1+\epsilon}} |\lambda_{n_j}| \geq \frac{1}{j^{1+\epsilon}} j \geq \frac{1}{j^{\epsilon}}$$

$$\not\longrightarrow 0$$

$\therefore \quad x_n \not\rightharpoonup.$

ⓒ Suppose $U \subseteq \mathcal{H}$ is a weakly open nbd of $0 \in \mathcal{H}$.

$\exists$ finitely many $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \in \mathcal{H}^*$

s.t.

$$0 \in V = \{ x \in \mathcal{H} \mid \underbrace{|\Lambda_i(x)| < 1}_{|\langle v_i, x_j \rangle| < 1} \ \forall i = 1, \ldots, n \} \subseteq U.$$

If a topology is generated by a given set S then open set in that topology

= arbitrary union of finite intersections of elements S.

By Riesz rep. thm

$$\Lambda_i(x) = \langle v_i, x\rangle \text{ for some } v_i \in \mathcal{H} \quad \forall i = 1, \ldots, n$$

Then

$$\sum_{i=1}^{n} \|v_i\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{\infty} |\langle v_i, e_j\rangle|^2$$

$$= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{n} |\langle v_i, e_j\rangle|^2 \right) < \infty$$

$\Rightarrow$ there must exist infinitely many $j \in \mathbb{N}$ s.t.

$$\sum_{i=1}^{n} |\langle v_i, e_j\rangle|^2 \leq \frac{1}{2n}$$

Otherwise the above series will diverge.

Then $|\lambda_j| \leq \sqrt{j} \Rightarrow$

$$|\Lambda_i(x_j)| = |\langle v_i, \lambda_j e_j\rangle| = |\lambda_j| |\langle v_i, e_j\rangle|$$

$$\leq \sqrt{j} \cdot \frac{1}{\sqrt{2j}} = \frac{1}{\sqrt{2}} < 1$$

$\forall\ i = 1, \ldots, n.$

infinitely many j

$\Rightarrow \quad x_j \in V$ for infinitely many j's $\boxed{}$

(5)

Problem 5

Find a sequence $f_n \in L^p(\mathbb{R})$ for $1 < p < \infty$ that converges weakly to 0 but satisfies $\|f_n\|_{L^p} = 1$ for all n, and deduce that f_n has no L^p-convergent subsequence.

$$f_n(t) = \begin{cases} e^{int}, & t \in (-\pi, \pi) \\ 0, & \text{Otherwise} \end{cases}$$

Clearly, $\|f\|_{L^p} = 1$.

Use the density of polynomials in L^p to note that for any $f \in L^p$, $\exists$ a polynomial p s.t.

$$\|f - p\|_{L^2} < \epsilon.$$

Also note that, as $n \to \infty$

$$\langle f_n, \phi \rangle_{L^2} \longrightarrow 0$$