Problem Set 6

1. $f : I \longrightarrow V$ is a function.

(a) f is Lipschitz-continuous $\iff$ $\exists$ $a \in I$ and $v_0 \in V$ and $g \in L^\infty(I)$ w/ $\|g\|_{L^\infty} \leq C$ s.t

$$f(x) = v_0 + \int_a^x g(t)\, dt \qquad \forall\ x \in I.$$

$\Leftarrow$

$$|f(x) - f(y)| = \left| \int_y^x g(t)\, dt \right| \leq \int_y^x |g(t)|\, dt$$

$$\leq C\,|x-y|$$

$\Rightarrow$ f is Lipschitz.

$\Rightarrow$ It's enough to prove that f is absolutely continuous.

$\forall\ \epsilon > 0$ $\exists$ $\delta > 0$ s.t. , $[a_i, b_i]$, $a_1 \leq b_1 \leq a_2 \leq \dots \leq a_n \leq b_n$

$$\sum_{j=1}^n |b_j - a_j| < \delta \implies \sum_{j=1}^n |f(b_j) - f(a_j)| < \epsilon.$$

$\because$ f is Lipschitz $\Rightarrow$ $\exists$ $C > 0$ s.t. $|f(x) - f(y)| \leq C|x-y|$

Given $\epsilon > 0$, choose $\delta = \dfrac{\epsilon}{C}$.

If $\sum |b_j - a_j| < \delta$ then

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| \leq \sum_{j=1}^{n} C\,|b_j - a_j| \leq \epsilon$$

$\Rightarrow$ f is absolutely continous.

$\boxtimes$

b) $f : [0,1] \to \mathbb{R}$ s.t. f is absolutely cont but not Lipschitz.

$f(x) = \sqrt{x}$ on $[0,1]$.

$= \int \frac{1}{2\sqrt{x}}\,dx \to$ absolutely continuous.

f is not Lipschitz, $x \neq 0$

$$\frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{x} - 0}{x - 0} = \frac{1}{\sqrt{x}} \longrightarrow \infty \text{ at } x \to 0.$$

$[0,1]$

$\boxtimes$

c) $f \in L^1_{loc}(I)$ $F(x) = \int_a^x f(t)\,dt$ $a \in I$.

f has jump discontinuity at $x_0 \in I$.

Want :- F is NOT differentiable.

We have $[a,b] \ni x_0 \Rightarrow f \in L^1([a,b])$.

Suppose $f(x) \to L$ as $x \to x_0^+$

$\epsilon > 0$

$$|f(t) - L| < \epsilon \quad \text{whenever } 0 < t - x_0 < \delta$$

$\Rightarrow$ for $0 < t - x_0 < \delta$

$$L - \epsilon < f(t) < L + \epsilon$$

Integrating the above from x_0 to $x_0 + h$ where $0 < h < \delta$, we get

$$h(L - \epsilon) < \int_{x_0}^{x_0+h} f(t)\, dt < h(L + \epsilon)$$

$$\Rightarrow \quad L - \epsilon < \frac{F(x_0+h) - F(x_0)}{h} < L + \epsilon$$

$\Rightarrow$ as $h \to 0$ we get

$$\text{RHD of } F = L = \text{RHL of } f(x)$$

Doing the same thing LHL

$$\text{LHD of } F = L' = \text{LHL of } f(x)$$

$\because \; L \neq L' \Rightarrow$ RHD of $F \neq$ LHD of F

$\Rightarrow$ F is not differentiable at x_0.

$\boxed{}$

(d) $\varphi = \chi_{[0,\infty)} : \mathbb{R} \to \mathbb{R}$ $\Rightarrow$ $\varphi(x) = 1$ if $x \in [0,\infty)$
$= 0$ otherwise

$\{q_i\}_{i=1}^{\infty}$ is an enumeration of $\mathbb{Q}$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi(x - q_n)$$

<u>Want</u> $f \in L^{\infty}(\mathbb{R})$ s.t. $\lim_{x \to q_n^+} f(x) = f(q_n)$

$\lim_{x \to q_n^-} f(x) = f(q_n) - \frac{1}{2^n}$

$\because$ $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series $\Rightarrow$ converges

and $f \leq \sum_{n=1}^{\infty} \frac{1}{2^n}$ $\Rightarrow$ $f \in L^{\infty}$.

note
$\varphi(h) = 1$ for $h \geq 0$
$\varphi(-h) = 0$, $h > 0$

$$f(q_n) = \sum_{m=1}^{\infty} \frac{1}{2^m} \varphi(q_n - q_m).$$

$$= \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{2^m} \varphi(q_n - q_m) + \frac{1}{2^n}$$

now $\displaystyle \lim_{x \to q_n^+} f(x) = \lim_{h \to 0} \sum_{m=1}^{\infty} \frac{1}{2^m} \varphi(q_n + h - q_m)$

$$= \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{2^m} \varphi(q_n - q_m) + \frac{1}{2^n} \varphi(h)$$

$$= \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{2^m} \varphi(q_n - q_m) + \frac{1}{2^n}$$

$$= f(q_n)$$

$$\lim_{x \to q_n^-} f(x) = \lim_{h \to 0} \sum_{m=1}^{\infty} \frac{1}{2^m} \varphi(q_n - h - q_m)$$

$$= \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{2^m} \varphi(q_n - q_m) + \underbrace{\frac{1}{2^n} \varphi(-h)}_{0}$$

$$= f(q_n) - \frac{1}{2^n}$$

e) F which is Lipschitz s.t. F is not differentiable on a dense subset of $\mathbb{R}$.

from d), choose $F = \int f(x)\, dx$ where $f(x)$ is the function in part d)

$\because$ $f \in L^\infty \Rightarrow$ by part a) F is Lipschitz.

by part c) $\because$ f has a jump discontinuity at $q_n \in \mathbb{Q} \Rightarrow F = \int f(x)\,dx$ is NOT differentiable at any rational number $\Rightarrow$ on a dense subset of $\mathbb{R}$.

$\blacksquare$

$m([0,1]) = m([0,1] \cap I)$

2) a) $\chi_{\mathbb{Q}^n}$ b) $\chi_{\mathbb{R}^n \setminus \mathbb{Q}^n}$

if $x \in X$ is a Lebesgue point

$$\lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\,dy = f(x) \quad \text{--- (1)}$$

$\int_A 1\,dm = m(A)$

a) if $x \in \mathbb{Q}^n$

RHS of (1) $= 1$, LHS $= 0$

$\Rightarrow$ $x \in \mathbb{Q}^n$ cannot be a Lebesgue point.

$x \in \mathbb{R}^n \setminus \mathbb{Q}^n$ is a Lebesgue point.

$\Rightarrow$ $\mathbb{R}^n \setminus \mathbb{Q}^n$ is the set of Lebesgue points of $\chi_{\mathbb{Q}^n}$.

b) $\chi_{\mathbb{R}^n \setminus \mathbb{Q}^n}$.

if $x \in \mathbb{Q}^n$, it is NOT a Lebesgue point.

$x \in \mathbb{R}^n \setminus \mathbb{Q}^n$ is a Lebesgue point.

$\mathbb{R}^n \setminus \mathbb{Q}^n$ is the set of Lebesgue points.

$\boxtimes$

3) $\mathbb{D}^n \subset \mathbb{R}^n$ denote the unit ball

$$f(x) = \frac{1}{|x|^\alpha}, \quad \mathbb{R}^n \setminus \{0\} \quad \alpha > 0.$$

a) $f \in L^1_{weak}(\mathbb{D}^n)$, $f \in L^1(\mathbb{D}^n)$.

$$\mu\{x \in \mathbb{D}^n \mid |f(x)| > t\} \leq \frac{C}{t} \quad \forall t > 0.$$

Use polar coordinates on $\mathbb{D}^n$

$$f(x) = \frac{1}{r^\alpha}, \quad r = |x|$$

$$m\left(x \in \mathbb{D}^n \mid \frac{1}{r^\alpha} > t\right) \leq \frac{C}{t} \quad \forall t > 0$$

$$\Rightarrow m\left(0 < r < 1 \mid \frac{1}{r^\alpha} > t\right) \leq \frac{C}{t} \quad \forall t > 0$$

$\frac{1}{r^\alpha} > t \Rightarrow r^\alpha < \frac{1}{t}$

$\Rightarrow r < \frac{1}{t^{1/\alpha}}$

if $0 < t < 1 \Rightarrow r < \frac{1}{t^{n\alpha}}$: $\forall \alpha > 0$

because $r \in [0,1)$

& $\frac{1}{t^{n\alpha}} > 1$.

In the case $0 < t < 1$, any $\alpha > 0$ is possible

case:- $t \geq 1$

$$m\left(x \in \mathbb{D}^n \mid \frac{1}{r^\alpha} > t\right) \leq \frac{C}{t}$$

Want $\frac{K}{t^{n/\alpha}} \leq \frac{C}{t}$ where K comes from the volume of S^n

will happen only when $\frac{n}{\alpha} \geq 1$

$\Rightarrow n \geq \alpha \Rightarrow \alpha \in (0, n]$

$\therefore$ $\alpha \in (0, n]$ for $f \in L^1_{weak}(\mathbb{D}^n)$.

Want. $f \in L^1(\mathbb{D}^n)$. integrating $f = \frac{1}{r^\alpha}$ i.e. polar coordinates

$$\int_0^1 \int_{S^{n-1}} \frac{1}{r^\alpha} r^{n-1} \, d\theta^{n-1} \, dr$$

$$= (K\pi)^{n-1} \int_0^1 r^{n-1-\alpha} \, dr$$

$$= (K\pi)^{n-1} \left[\frac{r^{n-\alpha}}{n-\alpha} \right]_0^1$$

will be finite if $n-\alpha > 0$

$\Rightarrow n > \alpha \Rightarrow \alpha \in (0, n)$.

(b) $\alpha \in [n, \infty)$ for $L^1_{weak}(\mathbb{R}^n \setminus \mathbb{D}^n)$

$\alpha \in (n, \infty)$ for $L^1(\mathbb{R}^n \setminus \mathbb{D}^n)$

4) $\lambda \ll \mu$

We prove by contradiction.

Suppose $\exists \; \epsilon > 0$ s.t. $\forall \; n = 1, 2, 3, \ldots$

$\exists \; E_n \subset X$ s.t. $\mu(E_n) < \frac{1}{2^n}$ but $\lambda(E_n) \geq \epsilon$.

Let $F_K = \bigcup_{n=K}^{\infty} E_n$ and $F = \bigcap_{K=1}^{\infty} F_K$

$$\mu(F_K) \leq \sum_{n=K}^{\infty} \frac{1}{2^n} = \frac{1}{2^{K-1}}$$

$\Rightarrow \quad \mu(F) = 0 \quad$ but $\quad \lambda(F_k) > \epsilon \quad \forall k$

$\because \; \lambda$ is finite $\Rightarrow \lambda(F) > \epsilon$.

$\Rightarrow \quad \mu(F) = 0 \quad$ But $\quad \lambda(F) > \epsilon$

contradictic $\quad \lambda \ll \mu$.

$\square$

5 $\quad \displaystyle\int_X g \, d\lambda = \int_X hg \, d\lambda + \int_X hg \, d\mu$

$h : X \to \mathbb{R}$, $\quad 0 \leq h < 1$ on $X \setminus E$, $\mu(E) = 0$

$f = \dfrac{h}{1-h} : X \setminus E \to [0, \infty)$

$\displaystyle\int_A f \, d\mu \leq \lambda(A)$

a) $\quad X = \mathbb{R}^n$, $\mu = m$, $\lambda = \delta$

$\displaystyle\int_{\mathbb{R}^n} g \, d\delta = \int_{\mathbb{R}^n} hg \, d\delta + \int_{\mathbb{R}^n} hg \, dm$

$\displaystyle g(0) = h(0) g(0) + \int_{\mathbb{R}^n} hg \, dm$

If $\quad h(0) = 1 \quad$ and $\quad h \equiv 0 \quad$ a.e. on $\mathbb{R}^n$

$\therefore \quad f = \dfrac{h}{1-h} \implies f = 0$ on $\mathbb{R}^n \setminus \{0\}$

$f = \infty$ on 0

$\int_A f \, dm = \delta(A)$ (want)

$\Rightarrow \quad 0 = \delta(A) \implies A \not\ni 0$

If $\mu(B) = 0 \implies \lambda(B) = 0$

$m(B) = 0 \implies \delta(B) = 0$

False as if $B = \{0\} \implies m(B) = 0$ but $\delta(B) = 1$.

$\lambda \not\ll \mu$.

(b) $\mu = \delta$, $\lambda = m$

$\int g \, dm = \int hg \, dm + \int hg \, d\delta$

$\Rightarrow \int g \, dm = \int hg \, dm + h(0) g(0)$

$h = 1$ a.e.

$h(0) = 0$ $\implies f = \dfrac{h}{1-h}$

$= 0$ at 0

$= \infty$ a.e.

$$\int_A f\,d\delta = m(A) \Rightarrow f(0) = m(A)$$
$$\Rightarrow 0 = m(A)$$

$\Rightarrow$ all sets A w/ $m(A) = 0$.

$\lambda \not\ll \mu$ b/c $\mu\left((0,1)\right) = 0$

But $m\left((0,1)\right) = 1$

c) $X = \mathbb{Z}^n$, $\mu = \nu$, $\lambda = \delta$

$$\int g\,d\delta = \int hg\,d\delta + \int hg\,d\nu$$

$$\Rightarrow g(0) = h(0)g(0) + \sum_{x \in \mathbb{Z}^n} h(x)g(x)$$

$$\Rightarrow g(0) = h(0)g(0) + h(0)g(0) + \sum_{\substack{x \in \mathbb{Z}^n \\ x \notin \{0\}}} h(x)g(x)$$

$$\Rightarrow g(0) = 2h(0)g(0) + \sum_{x \neq \{0\}} h(x)g(x)$$

$h = 0 \quad \forall x \notin 0$ $\Rightarrow f = \dfrac{h}{1-h}$

$h(0) = \dfrac{1}{2}$

$= 0 \quad \forall x \in \mathbb{Z}^n \setminus \{0\}$

$= 1$ on $x = 0$

$f = \chi_{\{0\}}$.

$\int_A f \, d\nu = \delta(A) \Rightarrow \sum_{a \in A} f(a) = \delta(A)$

$\Rightarrow f(0) + 0 = \delta(A)$

$\Rightarrow 1 = \delta(A)$

$\therefore \int_A f d\nu = \delta(A)$ for all $A \subset \mathbb{Z}^n$.

$\lambda \ll \mu$ if $\nu(B) = 0 \Rightarrow B = \emptyset$

$\Rightarrow \delta(B) = 0$

d)

$\int g \, d\nu = \int hg \, d\nu + \int hg \, d\delta$

$\Rightarrow \sum_{x \in \mathbb{Z}^n} g(x) = \sum h(x) g(x) + h(0) g(0)$

$f(0) = 1$

$f = \infty$ everywhere else.

equality holds ie $\int f d\delta = \nu(A)$

$\Rightarrow f(0) = \nu(A)$

$\Rightarrow 1 = \nu(A)$

$\Rightarrow \quad A \subset \{0\}.$

$\lambda \not\ll \mu.$

e) $\lambda = m, \quad \mu = \nu$

$\lambda \ll \mu \qquad \mu(A) = 0 \Rightarrow A = \emptyset$

$\Rightarrow m(A) = 0.$

$\mu = \nu$ is not σ-finite.

That is why anomaly is happening.

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