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Differentialgeometrie I

WiSe 2021–22

Problem Set 7

To be discussed: 7–8.12.2021

Problem 1

Prove: For each $k \geqslant 0$, a k-form $\omega \in \Omega^k(M)$ is closed if and only for every compact oriented $(k+1)$-dimensional submanifold $L \subset M$ with boundary, $\int_{\partial L} \omega = 0$.

Problem 2

Prove: On S^1, a 1-form $\lambda \in \Omega^1(S^1)$ is exact if and only if $\int_{S^1} \lambda = 0$.
Hint: Try to construct a primitive $f : S^1 \to \mathbb{R}$ by integrating λ along paths.

Problem 3

Suppose $\mathcal{O}$ is an open subset of either $\mathbb{H}^n$ or $\mathbb{R}^n$. We call $\mathcal{O}$ a *star-shaped* domain if for every $p \in \mathcal{O}$, it also contains the points $tp \in \mathbb{R}^n$ for all $t \in [0,1]$. It follows that $h(t,p) := tp$ defines a smooth homotopy $h : [0,1] \times \mathcal{O} \to \mathcal{O}$ between the identity and the constant map whose value is the origin, making $\mathcal{O}$ smoothly contractible. Use this homotopy to produce an explicit formula for a linear operator $P : \Omega^k(\mathcal{O}) \to \Omega^{k-1}(\mathcal{O})$ for each $k \geqslant 1$ satisfying

$$\omega = P(d\omega) + d(P\omega)$$

for all $\omega \in \Omega^k(\mathcal{O})$. In particular, whenever ω is a closed k-form, $P\omega$ is a primitive of ω.
Hint: Start with the chain homotopy that we constructed in lecture for proving the homotopy invariance of de Rham cohomology. As a sanity check, the answer to this problem can be found at the end of Lecture 13 in the notes, but try to find it yourself first.

Problem 4

Show that the wedge product descends to an associative and graded-commutative product $\cup : H^k_{\mathrm{dR}}(M) \times H^\ell_{\mathrm{dR}}(M) \to H^{k+\ell}_{\mathrm{dR}}(M)$, defined by

$$[\alpha] \cup [\beta] := [\alpha \wedge \beta].$$

This is called the *cup product* on de Rham cohomology.
Remark: There is similarly a cup product on singular cohomology, to which this one is isomorphic via de Rham's theorem. But this one is easier to define, and is thus often used in practice as a surrogate for the singular cup product.

Problem 5

For this exercise, identify the n-torus $\mathbb{T}^n$ with the quotient $\mathbb{R}^n/\mathbb{Z}^n$ (recall from Problem Set 2 #1 that there is a natural diffeomorphism). For any sufficiently small open set $\widetilde{\mathcal{U}} \subset \mathbb{R}^n$, the usual Cartesian coordinates $x^1, \dots, x^n : \widetilde{\mathcal{U}} \to \mathbb{R}$ can be used to define a smooth chart $(\mathcal{U}, x)$ on $\mathbb{T}^n$ where

$$\mathcal{U} := \left\{ [p] \in \mathbb{T}^n \;\middle|\; p \in \widetilde{\mathcal{U}} \right\}, \qquad x([p]) := (x^1(p), \dots, x^n(p)) \text{ for } p \in \widetilde{\mathcal{U}}.$$

(a) Show that the coordinate differentials $dx^1, \dots, dx^n \in \Omega^1(\mathcal{U})$ arising from the chart $(\mathcal{U}, x)$ described above are independent of the choice of the set $\widetilde{\mathcal{U}} \subset \mathbb{R}^n$, i.e. the definitions of the coordinate differentials obtained from two different choices $\widetilde{\mathcal{U}}_1, \widetilde{\mathcal{U}}_2 \subset \mathbb{R}^n$ coincide on the region $\mathcal{U}_1 \cap \mathcal{U}_2 \subset \mathbb{T}^n$ where they overlap.

(b) As a consequence of part (a), the 1-forms $dx^1, \dots, dx^n \in \Omega^1(\mathbb{T}^n)$ are well-defined on the entire torus, and they are obviously locally exact and therefore closed, but they might not actually be exact since none of the coordinates $x^1, \dots, x^n$ admit smooth definitions globally on $\mathbb{T}^n$. Show in fact that for any constant vector $(a_1, \dots, a_n) \in \mathbb{R}^n \backslash \{0\}$, the 1-form
$$\lambda := a_i \, dx^i \in \Omega^1(\mathbb{T}^n)$$
is closed but not exact.
Hint: You only need to find one smooth map $\gamma : S^1 \to \mathbb{T}^n$ such that $\int_{S^1} \gamma^* \lambda \neq 0$.

(c) One can similarly produce closed k-forms $\omega \in \Omega^k(\mathbb{T}^n)$ for any $k \leqslant n$ by choosing constants $a_{i_1 \dots i_k} \in \mathbb{R}$ and writing
$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(\mathbb{T}^n). \tag{1}$$
Show that for every nontrivial k-form of this type, one can find a cohomology class $[\alpha] \in H^{n-k}_{\mathrm{dR}}(\mathbb{T}^n)$ such that the cup product $[\omega] \cup [\alpha] \in H^n_{\mathrm{dR}}(\mathbb{T}^n)$ defined in Problem 4 is nontrivial, and deduce from this that ω is not exact.
Hint: Can you choose $\alpha \in \Omega^{n-k}(\mathbb{T}^n)$ so that $\omega \wedge \alpha$ is a volume form?

Remark: One can show that all cohomology classes in $H^k_{\mathrm{dR}}(\mathbb{T}^n)$ are representable by k-forms with constant coefficients as in (1), thus $\dim H^k_{\mathrm{dR}}(\mathbb{T}^n) = \binom{n}{k}$.

Problem 6
For V an n-dimensional vector space, the main goal of this exercise is to show that for every $v \in V$, the operator $\iota_v : \Lambda^* V^* \to \Lambda^* V^*$ defined by $\iota_v \omega := \omega(v, \cdot, \dots, \cdot)$ satisfies the graded Leibniz rule
$$\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_v \beta) \tag{2}$$
for all $\alpha \in \Lambda^k V^*$ and $\beta \in \Lambda^\ell V^*$. The statement is trivial if $v = 0$, so assume otherwise, in which case we may as well assume v is the first element e_1 of a basis $e_1, \dots, e_n \in V$, whose dual basis we can denote by $e^1_*, \dots, e^n_* \in V^* = \Lambda^1 V^*$.

(a) Prove that (2) holds whenever α and β are both products of the form $\alpha = e^{i_1}_* \wedge \dots \wedge e^{i_k}_*$ and $\beta = e^{j_1}_* \wedge \dots \wedge e^{j_\ell}_*$ with $i_1 < \dots < i_k$ and $j_1 < \dots < j_\ell$.
Hint: Consider separately a short list of cases depending on whether each of i_1 and j_1 are 1 and whether the sets $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_\ell\}$ are disjoint.

(b) Deduce via linearity that (2) holds always.

(c) Using (2), prove that for any manifold M and vector field $X \in \mathfrak{X}(M)$, the operator $P_X := d \circ \iota_X + \iota_X \circ d : \Omega^*(M) \to \Omega^*(M)$ satisfies the Leibniz rule
$$P_X(\alpha \wedge \beta) = P_X \alpha \wedge \beta + \alpha \wedge P_X \beta.$$
This is one of the main steps in a proof of Cartan's formula $\mathcal{L}_X \omega = P_X \omega$.

Problem 7
Prove that for any closed symplectic manifold (M, ω), $H^2_{\mathrm{dR}}(M)$ is nontrivial.
Hint: What can you say about the n-fold cup product of $[\omega] \in H^2_{\mathrm{dR}}(M)$ with itself?