Take-Home Midterm

Due: Wednesday, 26.01.2022 (100pts total)

Instructions

The purpose of this assignment is three-fold:

- In the absence of regular problem sets for the next two weeks, it deals with current material from the lectures (especially Problems 4 and 5).
- It gives the instructors a chance to gauge your understanding more directly than usual, and give feedback.
- It provides an opportunity to improve your final grade in the course.

Concerning the first point: if you are in the habit of working through the problem sets regularly, then we strongly recommend that you work through and hand in this one as well, even if you know you cannot solve enough problems to have an impact on your grade. This pertains especially to Problems 4 and 5, since they involve material that has not been covered on any problem sets so far.

To receive feedback and/or credit, you must either hand in your written solutions before the beginning of the lecture on **Wednesday, January 26 at 11:15**, or scan and e-mail them before that time to `wendl@math.hu-berlin.de`. The solutions will be discussed in the Übungen on that day.

You are free to use any resources at your disposal and to discuss the problems with your comrades, but **you must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

A score of 60 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus. The number of points assigned to each part of each problem is meant to be approximately proportional to its importance and/or difficulty.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. if you need to choose a smooth cutoff function with particular properties and can justify its existence with a convincing picture instead of an explicit formula (use your own judgement). Unless stated otherwise, you are free to make use of all results that have appeared in the lecture notes or in problem sets, without reproving them. There is one **exception**: some problems on this sheet may also have been stated as exercises in the lecture notes—in such cases (obviously), proof is required. When using a result from a problem set or the lecture notes, say explicitly which one.

If you get stuck on one part of a problem, it may often still be possible to move on and do the next part. You are free to ask for clarification or hints via e-mail/moodle or in office hours or Übungen; of course we reserve the right not to answer such questions.

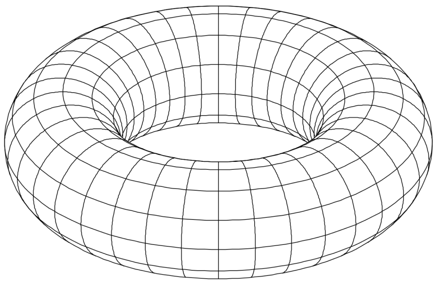
Problem 1 [20 pts]
In this problem we identify the 2-torus $\mathbb{T}^2$ with $\mathbb{R}^2/\mathbb{Z}^2$, so that points in $\mathbb{T}^2$ are written as equivalence classes $[(s,t)]$ for $(s,t) \in \mathbb{R}^2$, thus defining "coordinate" 1-forms $ds, dt \in \Omega^1(\mathbb{T}^2)$. (Note that these are not actually differentials since the coordinate functions s, t can be defined locally on neighborhoods of every point in $\mathbb{T}^2$ but not globally; in particular, ds and dt are closed but not exact—cf. Problem Set 7 #5.) We will consider the orientable 2-dimensional submanifold $\Sigma \subset \mathbb{R}^3$ defined as the image of the embedding

$$f : \mathbb{T}^2 \hookrightarrow \mathbb{R}^3 : [(s,t)] \mapsto ((2 + \cos 2\pi t)\cos 2\pi s, (2 + \cos 2\pi t)\sin 2\pi s, \sin 2\pi t)\,.$$

To see what Σ actually is, it helps to write $\theta := 2\pi s$, $\phi := 2\pi t$, and

$$f([(s,t)]) = 2\mathbf{v}(\theta) + (\cos\phi)\mathbf{v}(\theta) + (\sin\phi)\partial_z, \qquad \text{where} \qquad \mathbf{v}(\theta) := (\cos\theta, \sin\theta, 0) \in \mathbb{R}^3$$

and $\partial_z \in \mathbb{R}^3$ denotes as usual the standard basis vector in the z-direction. For each fixed s, the map $t \mapsto f([(s,t)])$ thus traces out the circle of radius 1 in the plane spanned by $\mathbf{v}(\theta)$ and ∂_z with center at $2\mathbf{v}(\theta)$. Here is a picture:



(a) [4 pts] Show that for each $\theta = 2\pi s$ and $\phi = 2\pi t$, the vector

$$\mathbf{n}(\theta,\phi) := (\cos\phi)\mathbf{v}(\theta) + (\sin\phi)\partial_z = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi) \in \mathbb{R}^3$$

is orthogonal to $T_{f([(s,t)])}\Sigma$.

(b) [8 pts] Assume Σ is endowed with the natural Riemannian metric that it inherits from the standard Euclidean inner product of $\mathbb{R}^3$, and $d\mathrm{vol} \in \Omega^2(\Sigma)$ denotes the resulting Riemannian volume form for some choice of orientation. Prove

$$f^* d\mathrm{vol} = \pm 4\pi^2\,(2 + \cos 2\pi t)\; ds \wedge dt,$$

where the sign $\pm$ depends on the choice of orientation. (Please do not worry about the orientation—I included "$\pm$" specifically so that you don't have to.)

(c) [8 pts] What is the area of the surface Σ?

Problem 2 [10 pts]
Recall that on a symplectic manifold (M,ω), any smooth function $H : M \to \mathbb{R}$ gives rise to a so-called *Hamiltonian* vector field, characterized uniquely by the condition $\omega(X_H, \cdot) = -dH$. The flow of X_H is symplectic, meaning it satisfies $(\varphi^t_{X_H})^*\omega = \omega$ for all t, or equivalently, $\mathcal{L}_{X_H}\omega \equiv 0$.

Assume $H^1_{\mathrm{dR}}(M) = 0$ and $Y \in \mathfrak{X}(M)$ is a vector field with a globally-defined flow that is symplectic ($\mathcal{L}_Y\omega \equiv 0$) and leaves H invariant, meaning $H \circ \varphi^t_Y \equiv H$ for all t. Show that

Y is then the Hamiltonian vector field for a function $F : M \to \mathbb{R}$ that is preserved under the flow of X_H, i.e. F is constant along all flow lines of X_H.

Comment: This is a simple case of a general principle known as Noether's theorem, which posits a bijective correspondence between symmetries of a mechanical system (such as the flow of the vector field Y) and quantities that are conserved under the motion of that system (e.g. the function F). Familiar examples of conserved quantities in mechanics are momentum, which corresponds to the invariance of the equations of motion under spatial translations, and angular momentum, which corresponds to rotational invariance.

Problem 3 [15 pts]
Let's do a little complex geometry. Fix an open subset $\mathcal{U} \subset \mathbb{C}$. For a smooth complex-valued function $f : \mathcal{U} \to \mathbb{C}$, the differential is a complex-valued 1-form $df \in \Omega^1(\mathcal{U}, \mathbb{C})$, i.e. if $f = u + iv$ for real-valued functions u and v, then $d_p f(X) = d_p u(X) + i d_p v(X) \in \mathbb{C}$ for each $p \in \mathcal{U}$ and $X \in T_p\mathcal{U} = \mathbb{C}$. Note that a complex-valued 1-form is only required to be *real*-linear at every point, not complex-linear. A good example is the conjugation map $f(x + iy) := x - iy$, whose differential is complex *anti*-linear at every point, i.e. $d_p f : T_p\mathcal{U} \to \mathbb{C}$ respects vector addition and satisfies $d_p f(\lambda X) = \bar{\lambda}\, d_p f(X)$ for all $X \in T_p\mathcal{U} = \mathbb{C}$.

Under the obvious bijection $\mathbb{C} = \mathbb{R}^2$ identifying $x+iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$, we can regard x and y as coordinates defining a chart on $\mathcal{U} \subset \mathbb{C}$, and associate to these the complex-valued functions $z = x + iy$ and $\bar{z} = x - iy$. Since x and y can also be written as functions of z and $\bar{z}$ in the form $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$, it is often useful to pretend that $(z, \bar{z})$ is an alternative "coordinate system" on $\mathcal{U}$, and express functions on $\mathcal{U}$ as functions of z and $\bar{z}$. In particular, for smooth functions $f : \mathcal{U} \to \mathbb{C}$, one can use a formal application of the chain rule to define

$$\begin{aligned} \partial_z f &= \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} := \frac{1}{2}\left(\partial_x f - i\partial_y f\right) \\ \partial_{\bar{z}} f &= \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} := \frac{1}{2}\left(\partial_x f + i\partial_y f\right). \end{aligned} \tag{1}$$

It should be stressed that this is not really a calculation, but rather a *definition*: z and $\bar{z}$ are fully determined by each other and thus cannot actually be understood as independent coordinates on $\mathbb{C}$, so that differentiating with respect to one while holding the other fixed makes no sense. Nonetheless, the exercises below make (1) seem like a sensible definition.

(a) [5 pts] Show that for any smooth function $f : \mathcal{U} \to \mathbb{C}$, $df = \partial_z f\, dz + \partial_{\bar{z}} f\, d\bar{z}$.

(b) [5 pts] Recall that a smooth function $f = u+iv : \mathcal{U} \to \mathbb{C}$ (with u and v assumed real-valued) is called *holomorphic* or *complex-analytic* if it satisfies the Cauchy-Riemann equations, which are the system of first-order PDEs $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$. Show that the following conditions are equivalent:

  (i) f is holomorphic on $\mathcal{U}$;
  (ii) $d_p f : T_p\mathcal{U} \to \mathbb{C}$ is complex-linear for every $p \in \mathcal{U}$;
  (iii) $\partial_{\bar{z}} f \equiv 0$.

(c) [5 pts] Assume $K \subset \mathcal{U} \subset \mathbb{C}$ is a compact 2-dimensional submanifold with boundary, and equip ∂K with the natural boundary orientation that it inherits from the standard orientation of $\mathbb{R}^2 = \mathbb{C}$. If $f : \mathcal{U} \to \mathbb{C}$ is holomorphic, what does Stokes' theorem[1] tell you about the integral $\int_{\partial K} f\, dz$?

Remark: The result of part (c) is a standard theorem of complex analysis, though in that context, one usually proves it without assuming that f is smooth, and then uses the result to prove that holomorphic functions are in fact always smooth. The latter can also be proved by other means, e.g. using the theory of elliptic PDEs.

Problem 4 [30 pts]
Throughout this problem, suppose M is a smooth manifold with a symmetric connection ∇ on its tangent bundle, and the associated tensor bundles $T^k_\ell M \to M$ are equipped with the connections naturally determined by ∇. Assuming $X, Y, Z \in \mathfrak{X}(M)$, prove:

(a) [5 pts] For any $\lambda \in \Omega^1(M)$, $d\lambda(X,Y) = (\nabla_X \lambda)(Y) - (\nabla_Y \lambda)(X)$.

(b) [5 pts] For any $\lambda \in \Omega^1(M)$, $(\mathcal{L}_X \lambda)(Y) = (\nabla_X \lambda)(Y) + \lambda(\nabla_Y X)$.

(c) [8 pts] For any $S \in \Gamma(T^0_2 M)$, $(\mathcal{L}_X S)(Y,Z) = (\nabla_X S)(Y,Z) + S(\nabla_Y X, Z) + S(Y, \nabla_Z X)$.
Hint: It suffices (why?) to verify this for tensor fields of the form $\lambda \otimes \mu \in \Gamma(T^0_2 M)$ with $\lambda, \mu \in \Omega^1(M)$. How does the operator $\mathcal{L}_X$ to behave under tensor products?

(d) [7 pts] Assume ∇ is the Levi-Cività connection for a pseudo-Riemannian metric $g = \langle\ ,\ \rangle$ on M. For each $p \in M$, g determines the so-called *musical isomorphisms*

$$T_p M \xrightarrow{\flat} T^*_p M : X \mapsto X_\flat := \langle X, \cdot \rangle, \qquad \text{and} \qquad T^*_p M \xrightarrow{\sharp := \flat^{-1}} T_p M : \lambda \mapsto \lambda^\sharp,$$

which similarly associate to each vector field $X \in \mathfrak{X}(M)$ a 1-form $X_\flat \in \Omega^1(M)$. Show that for $X \in \mathfrak{X}(M)$, the type $(0,2)$ tensor field $\nabla(X_\flat) \in \Gamma(T^0_2 M)$ is *antisymmetric* (i.e. it is a differential 2-form) if and only if X satisfies the relation

$$\langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle \equiv 0 \qquad \text{for all } Y, Z \in \mathfrak{X}(M). \tag{2}$$

Equation (2) is known as the *Killing equation*.

(e) [5 pts] Assuming M is compact, what can you say about the flow of a vector field $X \in \mathfrak{X}(M)$ if X satisfies the Killing equation (2)?

Problem 5 [25 pts]
One of the standard examples of "non-Euclidean" geometry is a Riemannian manifold known as the *Poincaré half-plane* $(\mathbb{H}, h)$. It is the smooth 2-manifold $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the Riemannian metric $h \in \Gamma(T^0_2 \mathbb{H})$ defined by

$$h_{(x,y)}(X,Y) := \frac{1}{y^2} \langle X, Y \rangle_E, \qquad \text{for } X, Y \in T_{(x,y)}\mathbb{H} = \mathbb{R}^2,$$

where $\langle\ ,\ \rangle_E$ denotes the standard Euclidean inner product on $\mathbb{R}^2$.

(a) [10 pts] Show that a smooth path $\gamma(t) = (x(t), y(t)) \in \mathbb{H}$ is a geodesic on $(\mathbb{H}, h)$ if and only if it satisfies the following second-order system of ordinary differential

[1] Integration of complex-valued forms is defined in a completely straightforward way, i.e. if $\alpha, \beta \in \Omega^n(M)$ are real-valued n-forms on an oriented n-manifold and $A \subset M$, $\int_A (\alpha + i\beta) := \int_A \alpha + i \int_A \beta \in \mathbb{C}$. There are similarly straightforward definitions for the wedge product and exterior derivative of complex-valued forms.

equations:

$$\begin{aligned} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} &= 0 \\ \ddot{y} + \frac{1}{y}\left(\dot{x}^2 - \dot{y}^2\right) &= 0. \end{aligned} \tag{3}$$

Advice: I'm not sure whether there is a cleverer way to do this, but it is definitely doable by directly computing the Christoffel symbols. Try not to compute any more than you actually have to, e.g. remember that since the connection is symmetric, some of the Christoffel symbols determine some of the others.

(b) [5 pts] Show that for any constants $x_0 \in \mathbb{R}$ and $r > 0$, the equations (3) admit solutions of the form

$$(x(t), y(t)) = (x_0, y(t))$$

for some function $y(t) > 0$, as well as

$$(x(t), y(t)) = (x_0 + r\cos\theta(t), r\sin\theta(t)).$$

for some function $\theta(t) \in (0, \pi)$.

(c) [5 pts] Do you think that *all* of the geodesics on $(\mathbb{H}, h)$ have the form described in part (b)? Answer with a brief heuristic argument, preferably based on a picture.

(d) [5 pts] Prove that $\exp(Y)$ is well defined for all tangent vectors $Y \in T\mathbb{H}$ that point upward or downward, i.e. that are proportional to ∂_y.
Hint: What is the length of a geodesic of the form $\gamma(t) = (x_0, y(t))$?