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Differentialgeometrie I

WiSe 2024–25

Practice Final Exam / Probeklausur

This is a lightly edited version of an exam that was given in this course when I taught it in a previous year. It should give a roughly accurate impression of the level of difficulty for this year's exams. The real exam might be slightly shorter.

Instructions

For reference, you may use any notes or books that you bring with you, but nothing electronic, i.e. no calculators or smartphones.

All answers require justification (within reason) in order to receive full credit, though you need not reprove any results that were proved in the lectures or on the problem sets. Keep in mind that if you get stuck on one part of a problem, it may sometimes be possible to skip it and do the next part.

Problem 1 [10 pts]
Recall that the 2-dimensional torus is defined as $\mathbb{T}^2 := \mathbb{R}^2/\sim$, where the equivalence relation $\sim$ identifies any two vectors that differ by an element of $\mathbb{Z}^2$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denote the natural quotient projection. Then the maps

$$\varphi_\alpha : (0,1) \times (0,1) \xrightarrow{\pi} \mathbb{T}^2, \qquad \varphi_\beta : (-1/2, 1/2) \times (0,1) \xrightarrow{\pi} \mathbb{T}^2,$$

each defined by sending pairs (θ, ϕ) to their equivalence classes $[(\theta,\phi)] \in \mathbb{R}^2/\mathbb{Z}^2$, are both embeddings onto open subsets $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ respectively in $\mathbb{T}^2$, and we can regard their inverses $x_\alpha := \varphi_\alpha^{-1} : \mathcal{U}_\alpha \to \mathbb{R}^2$ and $x_\beta := \varphi_\beta^{-1} : \mathcal{U}_\beta \to \mathbb{R}^2$ as charts. Write down an explicit formula for the transition map $x_\beta \circ x_\alpha^{-1} : x_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to x_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Is it smooth and/or orientation preserving?
Advice: Start by describing the domain $x_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \subset \mathbb{R}^2$.

Problem 2 [24 pts total]
Let (r,θ) denote the standard polar coordinates on $\mathbb{R}^2$, related to the Cartesian coordinates (x,y) by $x = r\cos\theta$ and $y = r\sin\theta$. This defines smooth 1-forms dr and $d\theta$ on $\mathbb{R}^2\backslash\{0\}$. Assume $f : [0,\infty) \to \mathbb{R}$ is a smooth function such that the rotationally symmetric function defined in polar coordinates by

$$\mathbb{R}^2\backslash\{0\} \to \mathbb{R} : (r,\theta) \mapsto \frac{f(r)}{r^2}$$

has a smooth extension over the origin; for instance, one example of such a function is $f(r) = r^2$, though there are many more examples. Note that f must necessarily satisfy $f(0) = f'(0) = 0$. We now define a 1-form λ on $\mathbb{R}^2\backslash\{0\}$ by

$$\lambda = f(r)\, d\theta.$$

(a) [12 pts] Show that λ admits an extension over the origin to define a smooth 1-form on $\mathbb{R}^2$.
Hint: What does λ look like in Cartesian coordinates?

(b) [12 pts] Show that $d\lambda \in \Omega^2(\mathbb{R}^2)$ is then a volume form near the origin if and only if $f''(0) \neq 0$.

Problem 3 [15 pts]
Consider the sphere S^2 with a volume form $\mu \in \Omega^2(S^2)$, and let $\mathrm{Diff}(S^2, \mu)$ denote the group of orientation- and area-preserving diffeomorphisms, i.e. diffeomorphisms $\varphi : S^2 \to S^2$ that satisfy $\varphi^*\mu = \mu$. The vector space $\mathfrak{diff}(S^2, \mu)$ of so-called *infinitessimal area-preserving diffeomorphisms* is then defined to consist of all vector fields $X \in \mathfrak{X}(S^2)$ such that the time-t flow $\varphi_X^t : S^2 \to S^2$ belongs to $\mathrm{Diff}(S^2, \mu)$ for all t close to zero.[1] Show that every $X \in \mathfrak{diff}(S^2, \mu)$ satisfies

$$\mu(X, \cdot) = dH$$

for some smooth function $H : S^2 \to \mathbb{R}$, and that H uniquely determines X.
Remark: You may use without proof the fact that $H^1_{\mathrm{dR}}(S^2)$ is trivial.

Problem 4 [15 pts]
Suppose M is a smooth n-manifold and $\lambda \in \Omega^1(M)$ is nowhere zero, so its kernel

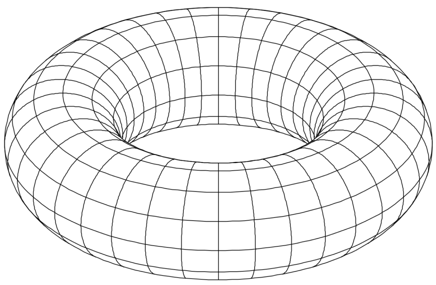
$$\xi := \ker \lambda := \{X \in TM \mid \lambda(X) = 0\} \subset TM$$

defines an $(n-1)$-dimensional distribution on M. Show that the following conditions are equivalent:

(i) Through every $p \in M$ there exists an $(n-1)$-dimensional submanifold everywhere tangent to ξ;

(ii) $d\lambda(X, Y) = 0$ for all pairs of vectors X and Y tangent to ξ.

Problem 5 [36 pts total]
Let $\Sigma \subset \mathbb{R}^3$ denote the surface obtained by rotating the circle $C_0 := \{(x, 0, z) \in \mathbb{R}^3 \mid (x-2)^2 + z^2 = 1\}$ about the z-axis. In other words, Σ is our usual picture of the torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$:



Assume Σ is endowed with the Riemannian metric g that it inherits from the standard Euclidean inner product on $\mathbb{R}^3$.

(a) [12 pts] For $i = 1, 2, 3$, consider the circles $C_i \subset \Sigma$ obtained by rotating the points

$$p_1 := (3, 0, 0), \; p_2 := (1, 0, 0), \; p_3 := (2, 0, 1) \in C_0$$

[1]The notation is motivated by thinking of $\mathrm{Diff}(S^2, \mu)$ as a Lie group with $\mathfrak{diff}(S^2, \mu)$ as the space of tangent vectors at the identity, i.e. its Lie algebra. This is difficult to make precise, however, since $\mathrm{Diff}(S^2, \mu)$ turns out to be infinite dimensional.

about the z-axis. Without any explicit computations, show that C_0, C_1 and C_2 are all images of geodesics on (Σ, g), but C_3 is not.
Hint: A helpful fact—which you may use without proof—is that every circle in $\mathbb{R}^3$ *can be parametrized by an embedding* $\gamma : S^1 \hookrightarrow \mathbb{R}^3$ *such that* $|\dot{\gamma}(t)|$ *is constant and* $\ddot{\gamma}(t)$ *always points from* $\gamma(t)$ *toward the center of the circle. How is* $\ddot{\gamma}(t)$ *related to* $T_{\gamma(t)}\Sigma$ *in each case, and why is this relevant?*

(b) [12 pts] If $\Sigma_+ \subset \Sigma$ denotes the portion of Σ lying in the region $\{x \geqslant 0,\ y \geqslant 0\}$ and $dA \in \Omega^2(\Sigma)$ is any choice of volume form such that $dA(X, Y) = 1$ for some orthonormal basis $X, Y \in T_p\Sigma$ at each $p \in \Sigma$, prove that the Gaussian curvature $K_G : \Sigma \to \mathbb{R}$ satisfies

$$\int_{\Sigma_+} K_G \, dA = 0.$$

Remark: I can think of at least a couple of ways to do this, using either version of the Gauss-Bonnet formula. Any correct argument will be accepted!

(c) [12 pts] Is $K_G(p_3)$ positive, negative, or zero? What about $K_G(p_1)$? Explain your answers briefly, but do not try to compute them explicitly.