



Problem Set 1

To be discussed: 23.10.2024

Problem 1

The standard *spherical coordinates* (r, θ, ϕ) on \mathbb{R}^3 are related to Cartesian coordinates by

$$x = r \cos \theta \cos \phi, \quad y = r \sin \theta \cos \phi, \quad z = r \sin \phi, \quad (1)$$

so θ plays the role of an angle in the xy -plane, and $\phi \in [-\pi/2, \pi/2]$ is the angle between the vector $(x, y, z) \in \mathbb{R}^3$ and the xy -plane.¹ Restricting to $r = 1$, the other two coordinates (θ, ϕ) can be used to describe points on the unit sphere $S^2 \subset \mathbb{R}^3$, though there are choices to be made since θ is only defined up to multiples of 2π (and it is not defined at all at the north and south poles $p_{\pm} := (0, 0, \pm 1) \in S^2$, where $\phi = \pm\pi/2$.)

- (a) Find two subsets $\mathcal{U}_1, \mathcal{U}_2 \subset S^2$ with $\mathcal{U}_1 \cup \mathcal{U}_2 = S^2 \setminus \{p_+, p_-\}$ such that for $i = 1, 2$, there are 2-dimensional charts of the form $(\mathcal{U}_i, \alpha_i)$ with $\alpha_i = (\theta_i, \phi_i)$, where the coordinate functions $\theta_i, \phi_i : \mathcal{U}_i \rightarrow \mathbb{R}$ are continuous and satisfy the spherical coordinate relations (1), and have images $\alpha_1(\mathcal{U}_1) = (0, 2\pi) \times (-\pi/2, \pi/2) \subset \mathbb{R}^2$ and $\alpha_2(\mathcal{U}_2) = (-\pi, \pi) \times (-\pi/2, \pi/2) \subset \mathbb{R}^2$.
- (b) One cannot use spherical coordinates to construct a chart on S^2 that contains either of the poles $p_{\pm} = (0, 0, \pm 1)$. Can you think of another way to construct charts on open subsets of S^2 that contain these two points?
Hint: On any sufficiently small neighborhood of p_+ or p_- in S^2 , every point has its z -coordinate determined by the x and y -coordinates.
- (c) Show that the charts you've constructed define a smooth atlas on S^2 . (If they don't, you might want to modify the charts you constructed in part (b).)

Problem 2

The *real projective n -space* \mathbb{RP}^n can be defined as the set of equivalence classes

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\}) / \sim,$$

where two nonzero vectors $v, w \in \mathbb{R}^{n+1}$ are considered equivalent if and only if $v = \lambda w$ for some $\lambda \in \mathbb{R}$. This means that each element of \mathbb{RP}^n is literally a line through the origin (excluding the origin itself) in \mathbb{R}^{n+1} .² It is convenient to denote points in \mathbb{RP}^n via so-called *homogeneous coordinates*, in which the symbol

$$[x_0 : \dots : x_n] \in \mathbb{RP}^n$$

means the equivalence class containing the vector $(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. Note that this symbol has no meaning unless at least one of the numbers x_0, \dots, x_n is nonzero. For

¹Achtung: there are various conventions for spherical coordinates in use. I'm told that this is the standard convention learned by mathematics students in Germany. I learned a different convention as a physics student in the USA.

²Since each line through the origin in \mathbb{R}^{n+1} contains exactly two unit vectors, which are antipodal points on S^n , one could alternatively define \mathbb{RP}^n as S^n / \sim for an equivalence relation on S^n such that every $x \in S^n$ is equivalent to its antipodal point $-x \in S^n$. This is the most popular way to define the *projective plane* \mathbb{RP}^2 , in particular.

$j = 0, \dots, n$, define $\mathcal{U}_j := \{[x_0 : \dots : x_n] \in \mathbb{R}\mathbb{P}^n \mid x_j \neq 0\}$ and a map $\varphi_j : \mathbb{R}^n \rightarrow \mathbb{R}\mathbb{P}^n$ by $\varphi_j(t_1, \dots, t_n) := [t_1 : \dots : t_j : 1 : t_{j+1} : \dots : t_n]$. Show that φ_j is an injective map onto \mathcal{U}_j , so $(\mathcal{U}_j, \varphi_j^{-1})$ is a chart, and compute the transition maps relating any two of the charts constructed in this way for different values of $j = 0, \dots, n$. Show that these $n + 1$ charts together form a smooth atlas, making $\mathbb{R}\mathbb{P}^n$ into a smooth n -dimensional manifold.

Problem 3

The *Klein bottle* can be defined as the set of equivalence classes $K^2 := \mathbb{R}^2 / \sim$, where \sim is the smallest equivalence relation on \mathbb{R}^2 such that $(s, t) \sim (s, t + 1)$ and $(s, t) \sim (s + 1, -t)$ for all $(s, t) \in \mathbb{R}^2$.

- (a) Do a google image search for “Klein bottle”, or simply look at Figure 6 in the lecture notes, and try to understand how the definition of K^2 stated above gives rise to such pictures.

Hint: If one makes a slight modification to the equivalence relation so that (s, t) is equivalent to $(s + 1, t)$ instead of $(s + 1, -t)$, one obtains an alternative definition of the 2-torus $\mathbb{T}^2 = S^1 \times S^1$. Think of the Klein bottle as a variation on this.

- (b) Show that K^2 is a smooth 2-dimensional manifold with a smooth atlas consisting of charts of the form (\mathcal{U}, x) where $\mathcal{U} := \{[(s, t)] \in K^2 \mid (s, t) \in \tilde{\mathcal{U}}\}$ for suitably chosen open subsets $\tilde{\mathcal{U}} \subset \mathbb{R}^2$ and $x([(s, t)]) = (s, t)$.

Problem 4

Consider the two 1-dimensional charts (\mathcal{U}, x) and (\mathcal{V}, y) on \mathbb{R} defined by $\mathcal{U} = \mathcal{V} := \mathbb{R}$, with $x(t) := t$ and $y(t) := t^3$.

- (a) Show that (\mathcal{U}, x) and (\mathcal{V}, y) are C^0 -compatible, but not C^1 -compatible.

Since both charts are globally defined, they each define atlases $\mathcal{A} := \{(\mathcal{U}, x)\}$ and $\mathcal{A}' := \{(\mathcal{V}, y)\}$ on \mathbb{R} . Taking the unique maximal smooth atlas containing \mathcal{A} gives the standard smooth structure on \mathbb{R} . If we instead take the unique maximal smooth atlas containing \mathcal{A}' , we obtain a different smooth structure on \mathbb{R} , and we will denote the smooth manifold obtained in this way by \mathbb{R}' : it is the same set as \mathbb{R} , but it carries a different smooth structure and is thus a different smooth manifold.

- (b) Show that the natural topologies induced on \mathbb{R} and \mathbb{R}' by their respective smooth structures are the same. Equivalently, the identity map $\mathbb{R} \rightarrow \mathbb{R}'$ is a homeomorphism.
- (c) Let $\mathcal{O} := (-1, 1) \subset \mathbb{R}'$, and show that the map

$$\varphi : \mathcal{O} \rightarrow \mathbb{R} : t \mapsto \tan\left(\frac{\pi}{2}t^3\right)$$

is a smooth chart on \mathbb{R}' .

- (d) A continuous map $f : M \rightarrow N$ between two smooth manifolds is called *smooth* if for every pair of smooth charts (\mathcal{U}, x) on M and (\mathcal{V}, y) on N , the map $y \circ f \circ x^{-1}$ is smooth wherever it is defined. Show that this holds for the identity map $\mathbb{R} \rightarrow \mathbb{R}'$, but *not* for its inverse. (In technical terms, this implies that the identity map $\mathbb{R} \rightarrow \mathbb{R}'$ is a homeomorphism, but not a diffeomorphism.)
- (e) Show that the map $\mathbb{R}' \rightarrow \mathbb{R} : t \mapsto t^2$ is not smooth.
- (f) Find an actual smooth bijection $\mathbb{R} \rightarrow \mathbb{R}'$ whose inverse is also smooth. (That is, \mathbb{R} and \mathbb{R}' are distinct smooth manifolds, but they are nonetheless *diffeomorphic*.)