



Problem Set 10

To be discussed: 8.01.2025

Notation: As in the lectures, \mathbb{F} denotes either \mathbb{R} or \mathbb{C} , and all vector spaces, vector bundles and linear maps are over \mathbb{F} unless otherwise specified. The dual of a vector space V is $V^* := \text{Hom}(V, \mathbb{F})$, and $\text{Hom}(V, W)$ denotes the space of linear maps $V \rightarrow W$.

Problem 1

- (a) Prove that for two finite-dimensional vector spaces V and W , there is a canonical isomorphism $\Psi : V^* \otimes W \rightarrow \text{Hom}(V, W)$ such that for all $\lambda \in V^*$ and $w \in W$, $\Psi(\lambda \otimes w)v = \lambda(v)w$.
- (b) Given smooth vector bundles E and F of rank m and k respectively over the same manifold M , describe a collection of smoothly compatible local trivializations of

$$\text{Hom}(E, F) := \bigcup_{p \in M} \text{Hom}(E_p, F_p),$$

giving $\text{Hom}(E, F)$ the structure of a smooth vector bundle of rank mk over M .

Hint: One can just as well describe local frames instead of trivializations.

- (c) Assume $A : E \rightarrow F$ is a map whose restriction $A_p := A|_{E_p}$ to the fiber $E_p \subset E$ over each point $p \in M$ is a linear map to the corresponding fiber $F_p \subset F$; in other words, the map $p \mapsto A_p$ is a section of the bundle $\text{Hom}(E, F)$. Show that it is a *smooth* section if and only if $A : E \rightarrow F$ is a smooth map between manifolds.

Remark: This shows that the notion of a “smooth linear bundle map $E \rightarrow F$ ” as we defined it in lecture is the same thing as a smooth section of $\text{Hom}(E, F)$.

- (d) Show that if $F \subset E$ is a smooth subbundle of the vector bundle $E \xrightarrow{\pi} M$, then the natural map $E \rightarrow E/F$ that restricts to each fiber $E_p \subset E$ as the quotient projection $E_p \rightarrow E_p/F_p : v \mapsto [v]$ is a smooth linear bundle map.

Problem 2

Assume (M, g) is a Riemannian n -manifold and $N \subset M$ is a smooth k -dimensional submanifold, so for every $p \in N$, $T_p N \subset T_p M$ is a linear subspace and has a well-defined orthogonal complement $(T_p N)^\perp \subset T_p M$ with respect to the inner product g_p . Prove:

- (a) $TN^\perp := \bigcup_{p \in N} (T_p N)^\perp$ is a smooth subbundle of $TM|_N$.
Hint: Construct smooth local frames X_1, \dots, X_n for $TM|_N$ such that X_1, \dots, X_k are tangent to N and X_{k+1}, \dots, X_n lie in $(TN)^\perp$.
- (b) The composition of the inclusion $TN^\perp \hookrightarrow TM|_N$ with the fiberwise quotient projection $TM|_N \rightarrow (TM|_N)/TN =: \nu N$ from Problem 1(d) defines a bundle isomorphism $TN^\perp \rightarrow \nu N$.

Problem 3

Prove:

- (a) A (real or complex) line bundle is trivial if and only if it admits a section that is nowhere zero.

- (b) A real vector bundle of any rank is orientable if and only if it admits a volume form.
- (c) A real line bundle is orientable if and only if it is trivial.
- (d) A real vector bundle $E \rightarrow M$ of rank m is orientable if and only if the bundle $\Lambda^m E \rightarrow M$ is trivial.

Problem 4

- (a) Prove that every real vector bundle is isomorphic to its dual bundle.
Hint: For a finite-dimensional vector space V , an isomorphism $V \rightarrow V^$ always exists but is not typically canonical. Your bundle isomorphism $E \rightarrow E^*$ will similarly need to depend on a non-canonical choice.*
- (b) Do you think every complex vector bundle is isomorphic to its dual? Just think about it—don't try to prove anything.

Problem 5

For subbundles $E^1, \dots, E^k \subset E$, we write $E = E^1 \oplus \dots \oplus E^k$ if the natural map

$$E^1 \oplus \dots \oplus E^k \rightarrow E : (v_1, \dots, v_k) \mapsto v_1 + \dots + v_k$$

is a bundle isomorphism. Suppose a splitting of this form exists, and write $m_i := \text{rank}(E^i)$ for $i = 1, \dots, k$ and $m := \text{rank}(E)$. What does the existence of this splitting tell you about the structure group of E , i.e. to what subgroup of $\text{GL}(m, \mathbb{F})$ can it be reduced?

Problem 6

Prove: If $E \rightarrow M$ is a real vector bundle with an indefinite bundle metric $\langle \cdot, \cdot \rangle$ of signature (k, ℓ) , then $E = E^+ \oplus E^-$ for a pair of subbundles $E^+, E^- \subset E$ of ranks k and ℓ respectively such that $\langle \cdot, \cdot \rangle$ is positive-definite on each fiber of E^+ and negative-definite on each fiber of E^- . Are the subbundles $E^\pm \subset E$ unique?