Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, C. S. Hsueh

Differentialgeometrie I
WiSe 2024–25

Problem Set 11

To be discussed: 15.01.2025

Notation: *As in the lectures, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$, and all vector spaces, vector bundles and linear maps are over $\mathbb{F}$ unless otherwise specified.*

Problem 1

Assume $\pi : E \to M$ is a smooth vector bundle with connection ∇, $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ and $\Phi_\beta : E|_{\mathcal{U}_\beta} \to \mathcal{U}_\beta \times \mathbb{F}^m$ are two smooth local trivializations related by transition functions $g_{\alpha\beta}, g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})$, and $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathbb{F}^{m\times m})$ and $A_\beta \in \Omega^1(\mathcal{U}_\beta, \mathbb{F}^{m\times m})$ denote the connection 1-forms associated to these two trivializations by ∇.

(a) Prove the "gauge transformation" formula:

$$A_\alpha(X) = g_{\alpha\beta}(p) A_\beta(X) g_{\beta\alpha}(p) + g_{\alpha\beta}(p)\, dg_{\beta\alpha}(X), \qquad \text{for } p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta,\ X \in T_pM.$$

Writing $g := g_{\beta\alpha}$ and $g^{-1} = g_{\alpha\beta}$, one often sees this relation abbreviated as

$$A_\alpha = g^{-1} A_\beta g + g^{-1}\, dg \qquad \text{on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

(b) Suppose $G \subset \mathrm{GL}(m, \mathbb{F})$ is a Lie subgroup, with Lie algebra $\mathfrak{g} = T_{\mathbb{1}}G$. Prove that if $g_{\beta\alpha}(p) \in G$ and $A_\beta(X) \in \mathfrak{g}$ for all $p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $X \in T_pM$, then both terms on the right hand side of the formula in part (a) lie in $\mathfrak{g}$.
Remark: This proves of course that $A_\alpha(X) \in \mathfrak{g}$ as well, though you could also have deduced that from a result stated in lecture—do you see how?

(c) Show that if the group G in part (b) is abelian, then the formula in part (a) becomes

$$A_\alpha(X) = A_\beta(X) + g_{\alpha\beta}(p)\, dg_{\beta\alpha}(X).$$

Hint: If G is abelian, then matrices in G also commute with matrices in $\mathfrak{g}$. (Why?)

(d) Assume further that G is the unitary group $\mathrm{U}(1)$, whose Lie algebra is $\mathfrak{u}(1) = i\mathbb{R} \subset \mathbb{C}^{1\times 1}$. Show that the imaginary-valued 2-forms dA_α and dA_β are then identical on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

Remark: The tangent bundle of an oriented Riemannian 2-manifold (Σ, g) has stucture group $\mathrm{SO}(2)$, which is naturally isomorphic to $\mathrm{U}(1)$. The result of part (d) shows that for any connection on $T\Sigma$ compatible with this $\mathrm{U}(1)$-structure, there is a globally-defined 2-form $\omega \in \Omega^2(\Sigma)$ that is related to all of the connection 1-forms $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, i\mathbb{R})$ on their respective domains by $\omega = i\, dA_\alpha$. We will later see that this 2-form encodes the Gaussian curvature of (Σ, g), and this observation is one step in the proof of the Gauss-Bonnet formula.

Problem 2

For any Lie subgroup $G \subset \mathrm{GL}(m, \mathbb{F})$ and a matrix $\mathbf{A} \in \mathfrak{g} = T_{\mathbb{1}}G$, show that the unique solution $\mathbf{\Phi}(t) \in \mathbb{F}^{m\times m}$ to the initial value problem

$$\dot{\mathbf{\Phi}}(t) = \mathbf{A}\mathbf{\Phi}(t), \qquad \mathbf{\Phi}(0) = \mathbb{1}$$

satisfies $\mathbf{\Phi}(t) \in G$ for all t.
Hint: Define a smooth vector field on G for which $\mathbf{\Phi}(t)$ is a flow line.
Remark: A mild generalization of this problem in which $\mathbf{A}$ is allowed to depend smoothly on t is used in proving the theorem that a connection is G-compatible if and only if its connection 1-forms are $\mathfrak{g}$-valued (cf. Exercises 20.13 and 20.14 in the notes).

Problem 3
Suppose M is a smooth manifold and ∇ is a connection on its tangent bundle $TM \to M$.

(a) Show that the bilinear map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by
$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$
defines a type $(1, 2)$ tensor field on M. It is called the *torsion* tensor, and the connection ∇ is called *symmetric* if $T \equiv 0$.

(b) Show that ∇ is symmetric if and only if for every smooth chart $(\mathcal{U}, (x^1, \ldots, x^n))$ on M, the Christoffel symbols $\Gamma^i_{jk} : \mathcal{U} \to \mathbb{R}$ defined by
$$\Gamma^i_{jk} := dx^i \left(\nabla_j(\partial_k)\right), \qquad i, j, k \in \{1, \ldots, n\}$$
satisfy $\Gamma^i_{jk} \equiv \Gamma^i_{kj}$.

Problem 4
Fix a smooth vector bundle $\pi : E \to M$ and a smooth manifold N.

(a) Show that if $f_0, f_1 : N \to M$ are two smoothly homotopic maps, then the pullback bundles $f_0^* E$ and $f_1^* E$ over N are isomorphic.
Hint: Given a smooth homotopy $h : [0, 1] \times N \to M$, choose a connection on the bundle $h^* E \to [0, 1] \times N$ and use parallel transport. (We will prove next week that every smooth vector bundle admits a connection—for now just assume this.)

(b) Show that if M is smoothly contractible, then the bundle $E \to M$ is trivial. It follows in particular that every vector bundle over $\mathbb{R}^n$ or a ball is trivial.

Remark: The statements in this problem are also true for topological vector bundles and continuous homotopies, without assuming any smoothness, but the proofs in that context are tricker because one cannot use connections.

Problem 5
A connection ∇ on a vector bundle $\pi : E \to M$ is called *flat* if for every $p \in M$ and $v \in E_p$, there exists a neighborhood $\mathcal{U} \subset M$ and a section $s \in \Gamma(E|_{\mathcal{U}})$ with $\nabla s \equiv 0$ and $s(p) = v$. Prove:

(a) For any *finite* subgroup $G \subset \mathrm{GL}(m, \mathbb{F})$, a G-structure on $E \to M$ determines a flat connection on $E \to M$, characterized by the property that it looks like the trivial connection in every G-compatible trivialization.

(b) Conversely, if M is compact, then every flat connection on $E \to M$ arises from a G-structure as in part (a) for some finite subgroup $G \subset \mathrm{GL}(m, \mathbb{F})$.
Hint: Cover M by finitely-many open subsets on which E admits frames $s_1, \ldots, s_m$ that satisfy $\nabla s_a \equiv 0$ for $a = 1, \ldots, m$. What can you deduce about the transition functions relating any two such frames?

(c) Show that if $\dim M = 1$, then every connection on $E \to M$ is flat.

Remark: We will see when we discuss curvature that non-flat connections always exist when $\dim M \geqslant 2$.