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**Differentialgeometrie I**

**WiSe 2024–25**

# Problem Set 12

To be discussed: 5.02.2025

**Problem 1**

This is essentially a repeat of Problem Set 5 #1, but using concepts and terminology that we did not yet have at our disposal back then. Any nowhere-vanishing 1-form $\alpha \in \Omega^1(M)$ on a 3-manifold $M$ defines a 2-dimensional distribution $\xi \subset TM$ by $\xi_p := \ker \alpha_p \subset T_pM$.

(a) Deduce from the Frobenius theorem that $\xi$ is an integrable distribution if and only if $d\alpha|_\xi$ vanishes, and that the latter is also equivalent to the condition $\alpha \wedge d\alpha \equiv 0$.

(b) Show that for $\alpha = f(x)\,dy + g(x)\,dz \in \Omega^1(\mathbb{R}^3)$ with smooth functions $f, g : \mathbb{R} \to \mathbb{R}$, $\xi$ is integrable if and only if the vector $(f'(x), g'(x)) \in \mathbb{R}^2$ is a scalar multiple of $(f(x), g(x))$ for each $x$, and in this case, one can also write $\xi = \ker(A\,dy + B\,dz)$ for some constants $A, B \in \mathbb{R}$. (The integral submanifolds are then easy to find: they form a family of parallel planes in $\mathbb{R}^3$.)

**Problem 2**

An integrable $k$-dimensional distribution $\xi \subset TM$ on an $n$-manifold $M$ determines a *foliation* of $M$, which one thinks of as a decomposition of $M$ into a smooth family of disjoint integral submanifolds: every point in $M$ belongs to a unique *leaf* of the foliation, meaning a maximal connected subset $L \subset M$ of the form $L = f(\Sigma)$ where $\Sigma$ is a $k$-manifold and $i : \Sigma \to M$ is an injective immersion satisfying $\operatorname{im}(T_pf) = \xi_p$ for all $p \in \Sigma$.

Consider the integrable 1-dimensional distribution $\xi$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined by $\xi = \ker(a\,dx + b\,dy)$ for some constant $(a, b) \in \mathbb{R}^2\backslash\{0\}$, where $x, y$ are the usual coordinates on $\mathbb{R}^2$, whose coordinate differentials descend to closed (but not exact) 1-forms on the quotient $\mathbb{T}^2$. Show that the leaves of the resulting foliation on $\mathbb{T}^2$ can be described as follows:

(a) If $a/b \in \mathbb{Q}$ or $b = 0$, they are compact submanifolds diffeomorphic to $S^1$.

(b) Otherwise, they are images of injective immersions $\mathbb{R} \to \mathbb{T}^2$ and are dense in $\mathbb{T}^2$. (The latter implies that they cannot be submanifolds.)

**Problem 3**

Suppose $\ell \to S^1$ is a non-orientable real line bundle over $S^1$ (as for instance in Problem Set 9 #4). Find a path $\gamma : [0, 1] \to S^1$ with $\gamma(0) = \gamma(1) =: p$ such that the parallel transport $P^t_\gamma : \ell_p \to \ell_p$ cannot be the identity map for any choice of connection $\nabla$.

**Problem 4**

Suppose $\nabla$ is a flat connection on a vector bundle $E \to M$.

(a) Show that for any smooth map $f : N \to M$, the pullback of $\nabla$ to a connection on $f^*E \to N$ is also flat.

(b) Show that if $\{\gamma_s : [0, 1] \to M\}_{s\in[0,1]}$ is a smooth family of paths with fixed end points $\gamma_s(0) = p$ and $\gamma_s(1) = q$ for all $s \in [0, 1]$, then the two parallel transport maps $P^1_{\gamma_0}, P^1_{\gamma_1} : E_p \to E_q$ are the same.
*Hint: Write $h(s, t) := \gamma_s(t)$ and use the fact that the pullback connection on $h^*E \to$*

$[0,1] \times [0,1]$ *is also flat. Can you construct global flat sections of* $h^*E$*? What will they look like on the subsets* $[0,1] \times \{0\}$ *and* $[0,1] \times \{1\}$*?*[1]

**Problem 5**
For a connection $\nabla$ on a vector bundle $E \to M$, verify the following properties of the Riemann tensor that were stated in lecture:

(a) The map $R(X,Y)v = \nabla_X \nabla_Y v - \nabla_Y \nabla_X v - \nabla_{[X,Y]} v$ is $C^\infty$-linear with respect to each of $X, Y \in \mathfrak{X}(M)$ and $v \in \Gamma(E)$.

(b) The components ${R^a}_{jkb}$ of $R$ with respect to a chart for $M$ and frame for $E$ over some subset $\mathcal{U} \subset M$ are related to the Christoffel symbols $\Gamma^a_{jk}$ by

$$ {R^a}_{jkb} = \partial_j \Gamma^a_{kb} - \partial_k \Gamma^a_{jb} + \Gamma^a_{jc}\Gamma^c_{kb} - \Gamma^a_{kc}\Gamma^c_{jb}. $$

**Problem 6**
Show that if $\nabla$ is compatible with a bundle metric $\langle\ ,\ \rangle$ on $E \to M$, then the Riemann tensor satisfies the antisymmetry relation

$$ \langle R(X,Y)v, w\rangle + \langle v, R(X,Y)w\rangle = 0. $$

*Hint: Given* $X, Y \in \mathfrak{X}(M)$ *and* $v, w \in \Gamma(E)$*, compute* $\left(\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X - \mathcal{L}_{[X,Y]}\right)(\langle v, w\rangle)$.

**Problem 7**
For a connection $\nabla$ on the bundle $\pi : E \to M$, prove:

(a) For any $v \in \Gamma(E) = \Omega^0(M,E)$ and $X, Y \in T_pM$ at a point $p \in M$, $d^2_\nabla v := d_\nabla(d_\nabla v) \in \Omega^2(M,E)$ satisfies $(d^2_\nabla v)(X,Y) = R(X,Y)v$.

(b) The connection $\nabla$ is flat if and only if the covariant exterior derivative operators $d_\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ for all $k \geqslant 0$ satisfy $d_\nabla \circ d_\nabla = 0$.

**Problem 8**
Suppose $\pi : E \to M$ has structure group $G \subset \mathrm{GL}(m,\mathbb{F})$ with Lie algebra $\mathfrak{g} \subset \mathbb{F}^{m \times m}$, $\nabla$ is a $G$-compatible connection, $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ is a $G$-compatible local trivialization and $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})$ is the corresponding connection 1-form, satisfying the formula $(\nabla_X v)_\alpha = \mathcal{L}_X v_\alpha + A_\alpha(X) v_\alpha$ for $X \in \mathfrak{X}(\mathcal{U}_\alpha)$ and $v \in \Gamma(E|_{\mathcal{U}_\alpha})$. We define the *local curvature 2-form* $F_\alpha \in \Omega^2(\mathcal{U}_\alpha, \mathbb{F}^{m\times m})$ in terms of the curvature 2-form $\Omega_K \in \Omega^2(M, \mathrm{End}(E))$ by $\left(\Omega_K(X,Y)v\right)_\alpha = F_\alpha(X,Y)v_\alpha$.

(a) Prove the formula $F_\alpha(X,Y) = dA_\alpha(X,Y) + [A_\alpha(X), A_\alpha(Y)]$, where the bracket on the right hand side denotes the matrix commutator $[\mathbf{A},\mathbf{B}] := \mathbf{AB} - \mathbf{BA}$.
*Hint: Use the Riemann tensor as a stand-in for* $\Omega_K$.

(b) If $\Phi_\beta : E|_{\mathcal{U}_\beta} \to \mathcal{U}_\beta \times \mathbb{F}^m$ is a second trivialization related to $\Phi_\alpha$ by the transition map $g = g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$, show that $F_\beta(X,Y) = gF_\alpha(X,Y)g^{-1}$.

(c) Show that if $G$ is abelian, then $F_\alpha = dA_\alpha$ and it is independent of the choice of trivialization, thus defining a global 2-form $F \in \Omega^2(M,\mathfrak{g})$. (It is sometimes also called the *curvature 2-form* of $\nabla$.)

[1]For the purposes of Problem 4, you are safe in pretending that $[0,1] \times [0,1]$ is a smooth manifold, rather than something exotic like a "manifold with boundary and corners". If this worries you, assume that the family of paths $\gamma_s : [0,1] \to M$ is defined for $s \in \mathbb{R}$ instead of just $s \in [0,1]$; this does not change the situation in any significant way.