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Differentialgeometrie I
WiSe 2024–25

Problem Set 5

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Problem 1

Suppose M is a 3-manifold and $\alpha \in \Omega^1(M)$ is nowhere zero, meaning that $\alpha_p \neq 0 \in \Lambda^1 T_p^* M$ for every $p \in M$, thus defining a 2-dimensional subspace $\xi_p := \ker \alpha_p \subset T_p M$ of each tangent space. The set $\xi := \bigcup_{p \in M} \xi_p \subset TM$ in this situation is called a *smooth 2-plane field* in M. We say that ξ is *integrable* if its defining 1-form α satisfies the condition $\alpha \wedge d\alpha \equiv 0$.

(a) Show that the integrability condition depends only on ξ and not on α, i.e. for any $\beta \in \Omega^1(M)$ that is also nowhere zero and satisfies $\ker \beta_p = \xi_p$ for all $p \in M$, $\alpha \wedge d\alpha \equiv 0$ if and only if $\beta \wedge d\beta \equiv 0$.
Hint: If $\ker \alpha_p = \ker \beta_p$, how are the two cotangent vectors $\alpha_p, \beta_p \in T_p^* M$ related?

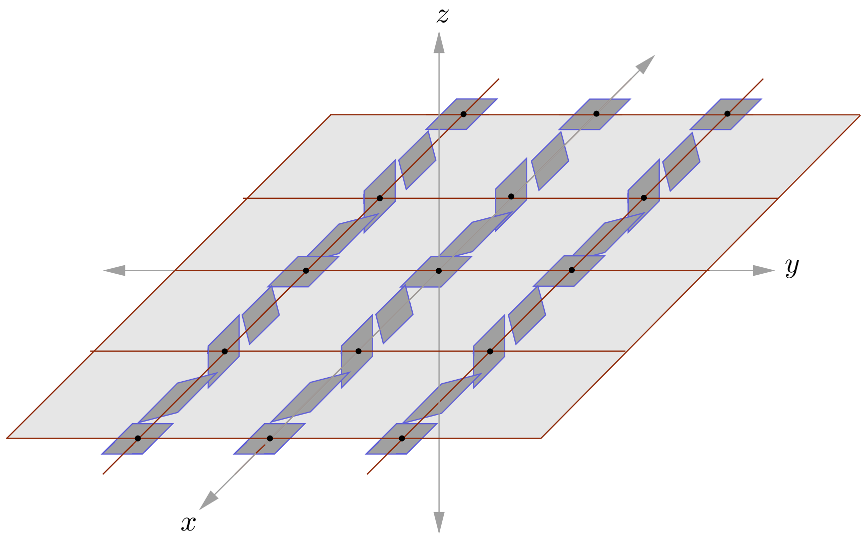
(b) Prove that the following conditions are each equivalent to integrability:

 (i) $(d\alpha)_p|_{\xi_p} \in \Lambda^2 \xi_p^*$ vanishes for every $p \in M$.
 Hint: Evaluate $(\alpha \wedge d\alpha)_p$ on a basis of $T_p M$ that includes two vectors in ξ_p.

 (ii) For every pair of vector fields $X, Y \in \mathfrak{X}(M)$ with $X(p), Y(p) \in \xi_p$ for all $p \in M$, $[X, Y] \in \mathfrak{X}(M)$ also satisfies $[X, Y](p) \in \xi_p$ for all $p \in M$.
 Hint: Use our original definition of the exterior derivative, via C^∞-linearity.

(c) Using Cartesian coordinates (x, y, z) on $M := \mathbb{R}^3$, suppose $\alpha = f(x)\, dy + g(x)\, dz$ for smooth functions $f, g : \mathbb{R} \to \mathbb{R}$. Under what conditions on f and g is ξ integrable? Show that if these conditions hold, then for every point $p \in \mathbb{R}^3$ there exists a 2-dimensional submanifold $\Sigma \subset \mathbb{R}^3$ such that $p \in \Sigma$ and $T_q \Sigma = \xi_q$ for all $q \in \Sigma$.

*Remark: The result of part (c) is a special case of the Frobenius integrability theorem, which we will prove later in this course. In case you're curious, the following picture gives an example of what $\xi \subset T\mathbb{R}^3$ might look like if it is **not** integrable. Can you picture a 2-dimensional submanifold that is everywhere tangent to ξ? (I didn't think so.)*



Problem 2
Prove: On an n-dimensional vector space V, a set of dual vectors $\lambda^1, \dots, \lambda^k \in V^* = \Lambda^1 V^*$ is linearly independent if and only if $\lambda^1 \wedge \dots \wedge \lambda^k \in \Lambda^k V^*$ is nonzero.
Hint: Consider products of the form $\left(\sum_{i=1}^k c_i \lambda^i\right) \wedge \lambda^2 \wedge \dots \wedge \lambda^k$.

Problem 3
For V an n-dimensional vector space, the main goal of this exercise is to show that for every $v \in V$, the operator $\iota_v : \Lambda^* V^* \to \Lambda^* V^*$ defined by $\iota_v \omega := \omega(v, \cdot, \dots, \cdot)$ satisfies the graded Leibniz rule

$$\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_v \beta) \tag{1}$$

for all $\alpha \in \Lambda^k V^*$ and $\beta \in \Lambda^\ell V^*$. The statement is trivial if $v = 0$, so assume otherwise, in which case we may as well assume v is the first element e_1 of a basis $e_1, \dots, e_n \in V$, whose dual basis we can denote by $e_*^1, \dots, e_*^n \in V^* = \Lambda^1 V^*$.

(a) Prove that (1) holds whenever α and β are both products of the form $\alpha = e_*^{i_1} \wedge \dots \wedge e_*^{i_k}$ and $\beta = e_*^{j_1} \wedge \dots \wedge e_*^{j_\ell}$ with $i_1 < \dots < i_k$ and $j_1 < \dots < j_\ell$.
Hint: Consider separately a short list of cases depending on whether each of i_1 *and* j_1 *are* 1 *and whether the sets* $\{i_1, \dots, i_k\}$ *and* $\{j_1, \dots, j_\ell\}$ *are disjoint.*

(b) Deduce via linearity that (1) holds always.

(c) Using (1), prove that for any manifold M and vector field $X \in \mathfrak{X}(M)$, the operator $P_X := d \circ \iota_X + \iota_X \circ d : \Omega^*(M) \to \Omega^*(M)$ satisfies the Leibniz rule

$$P_X(\alpha \wedge \beta) = P_X \alpha \wedge \beta + \alpha \wedge P_X \beta.$$

This is one of the main steps in a proof of Cartan's "magic" formula $\mathcal{L}_X \omega = P_X \omega$.

Problem 4
A *volume form* on an n-manifold M is an n-form $d\mathrm{vol} \in \Omega^n(M)$ that is nowhere zero, meaning that at every point $p \in M$, the tangent space $T_p M$ has a basis $X_1, \dots, X_n$ such that $d\mathrm{vol}(X_1, \dots, X_n) \neq 0$. If a volume form $d\mathrm{vol}$ is given, one can then define the *divergence* of a vector field $X \in \mathfrak{X}(M)$ with respect to $d\mathrm{vol}$ as the unique function $\mathrm{div}(X) : M \to \mathbb{R}$ such that the n-form $d(\iota_X d\mathrm{vol})$ matches $\mathrm{div}(X) \cdot d\mathrm{vol}$. Show that in local coordinates with respect to a chart $(\mathcal{U}, x)$ such that $d\mathrm{vol} = f\, dx^1 \wedge \dots \wedge dx^n$ for a function $f : \mathcal{U} \to \mathbb{R}$, the divergence is given in terms of the components X^i of X by

$$\mathrm{div}(X) = \frac{1}{f} \partial_i (f X^i) \qquad \text{on } \mathcal{U}.$$