HUMBOLDT-UNIVERSITÄT ZU BERLIN
Institut für Mathematik
C. Wendl, C. S. Hsueh

Differentialgeometrie I

WiSe 2024–25

Problem Set 6

To be discussed: 27.11.2024

Problem 1

(a) Find explicit oriented atlases for S^1 and S^2.

(b) Use the oriented atlases in part (a) to show that the antipodal map $S^n \to S^n : p \mapsto -p$ is orientation preserving for $n = 1$, but orientation reversing for $n = 2$.

(c) Without talking about atlases, prove that S^n is orientable for every $n \in \mathbb{N}$ by defining a continuous family of orientations of the tangent spaces $\{T_p S^n \mid p \in S^n\}$.
Hint: Any $p \in S^n$ together with a basis of $T_p S^n$ forms a basis of $\mathbb{R}^{n+1}$.

(d) Show that the antipodal map $S^n \to S^n$ is orientation preserving for every odd n and orientation reversing for every even n.

Problem 2

Recall that a diffeomorphism $\mathbb{R}^n \supset \mathcal{U} \xrightarrow{\psi} \mathcal{V} \subset \mathbb{R}^n$ is called orientation preserving if $\det D\psi(p) > 0$ for all $p \in \mathcal{U}$, and orientation reversing if $\det D\psi(p) < 0$ for all $p \in \mathcal{U}$. The fact that ψ is a diffeomorphism implies that for any fixed p, one of these conditions must hold, but it need not hold everywhere, i.e. not every diffeomorphism is either orientation preserving or orientation reversing.

(a) Show that if M is an oriented manifold, then every chart $(\mathcal{U}, x)$ whose domain $\mathcal{U} \subset M$ is connected is either orientation preserving or orientation reversing.

(b) In Problem Set 1 #3, we defined the Klein bottle as $K^2 := \mathbb{R}^2/\sim$, where $(s,t) \sim (s, t+1)$ and $(s,t) \sim (s+1, -t)$ for all $(s,t) \in \mathbb{R}^2$. Find a pair of charts $(\mathcal{U}_1, x_1)$ and $(\mathcal{U}_2, x_2)$ on K^2 such that the subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ are both connected but $\mathcal{U}_1 \cap \mathcal{U}_2$ has two connected components, and the transition map $x_1 \circ x_2^{-1}$ is neither orientation preserving nor orientation reversing.

(c) Explain why part (b) implies K^2 is not orientable.

(d) Find a continuous path $\gamma : [0,1] \to K^2$ with $\gamma(0) = \gamma(1) =: p$ and a continuous family of ordered bases $(X_1(t), X_2(t))$ of $T_{\gamma(t)} K^2$ such that $(X_1(0), X_2(0))$ and $(X_1(1), X_2(1))$ determine distinct orientations of the vector space $T_p K^2$.

Problem 3

Define a 1-form on $\mathbb{R}^2 \backslash \{0\}$ in the standard (x,y)-coordinates by $\lambda = \dfrac{1}{x^2 + y^2}(x\,dy - y\,dx)$, and let $i : S^1 \hookrightarrow \mathbb{R}^2$ denote the natural inclusion of the unit circle in $\mathbb{R}^2$.

(a) Using a finite covering by oriented charts and a partition of unity, compute $\int_{S^1} i^*\lambda = 2\pi$.

(b) Can you give the result of part (a) a nice interpretation in terms of polar coordinates?

(c) Show that for any smooth function $f : S^1 \to \mathbb{R}$, $\int_{S^1} df = 0$.

(d) Redo the computations of parts (a) and (c) without using a partition of unity: instead use a single chart whose domain covers all of S^1 except for a set of measure zero.

Problem 4

In local coordinates with respect to an oriented chart $(\mathcal{U}, x)$ on an oriented n-manifold M, a Riemannian metric $g \in \Gamma(T_2^0 M)$ is described in terms of its components $g_{ij} := g(\partial_i, \partial_j)$, so that vectors $X, Y \in T_pM$ at points $p \in \mathcal{U}$ satisfy $g(X, Y) = g_{ij} X^i Y^j$. The goal of this problem is to prove that the Riemannian volume form determined by g takes the form

$$d\text{vol} = \sqrt{\det \mathbf{g}}\, dx^1 \wedge \ldots \wedge dx^n \qquad \text{on } \mathcal{U}, \tag{1}$$

where $\mathbf{g} : \mathcal{U} \to \mathbb{R}^{n \times n}$ denotes the matrix-valued function whose ith row and jth column is g_{ij}. Note that this matrix necessarily has positive determinant since g is everywhere positive definite. Fix a point $p \in \mathcal{U}$ and a positively-oriented orthonormal basis $(X_1, \ldots, X_n)$ of T_pM, whose dual basis we will denote by $\lambda^1, \ldots, \lambda^n \in T_p^*M$. In this case, it was shown in lecture that $d\text{vol}_p = \lambda^1 \wedge \ldots \wedge \lambda^n$. Define matrices $\mathbf{X}, \boldsymbol{\lambda} \in \mathbb{R}^{n \times n}$ whose ith row and jth column are $dx^i(X_j)$ and $\lambda^i(\partial_j)$ respectively. By Proposition 9.10 in the lecture notes, $(\lambda^1 \wedge \ldots \wedge \lambda^n)(\partial_1, \ldots, \partial_n) = \det \boldsymbol{\lambda}$.

(a) Prove $\boldsymbol{\lambda} = \mathbf{X}^{-1}$.

(b) Prove $\mathbf{X}^T \mathbf{g} \mathbf{X} = \mathbb{1}$.

(c) Deduce (1).

Problem 5

Using Cartesian coordinates (x, y, z) on $\mathbb{R}^3$, let $\omega := x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \in \Omega^2(\mathbb{R}^3)$, and let $i : S^2 \hookrightarrow \mathbb{R}^3$ denote the inclusion of the unit sphere.

(a) Show that for an appropriate choice of orientation on S^2, $d\text{vol}_{S^2} := i^*\omega \in \Omega^2(S^2)$ is the Riemannian volume form corresponding to the Riemannian metric on S^2 that is induced by the Euclidean inner product of $\mathbb{R}^3$.
Hint: Pick a good vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ with which to write ω as $\iota_X(dx \wedge dy \wedge dz)$.

(b) Show that in the spherical coordinates (θ, ϕ) of Problem Set 1 #1, $d\text{vol}_{S^2} = \cos\phi\, d\theta \wedge d\phi$.

(c) On the open upper hemisphere $\mathcal{U}_+ := \{z > 0\} \subset S^2 \subset \mathbb{R}^3$, one can define a chart $(x, y) : \mathcal{U}_+ \to \mathbb{R}^2$ by restricting to $\mathcal{U}_+$ the usual Cartesian coordinates x and y, which are then related to the z-coordinate on this set by $z = \sqrt{1 - x^2 - y^2}$. Show that $d\text{vol}_{S^2} = \frac{1}{z}\, dx \wedge dy$ on $\mathcal{U}_+$.

(d) Compute the surface area of $S^2 \subset \mathbb{R}^3$ in two ways: once using the formula for $d\text{vol}_{S^2}$ in part (b), and once using part (c) instead. In both cases, you should be able to express the answer in terms of a single Lebesgue integral[1] over a region in $\mathbb{R}^2$, and there will be no need for any partition of unity.

[1] You may assume that the upper and lower hemispheres have the same area.