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Differentialgeometrie I
WiSe 2024–25

Problem Set 7

To be discussed: 4.12.2024

Problem 1
On an oriented Riemannian 3-manifold (M, g) with Riemannian volume form $d\mathrm{vol}$, the *curl* of a vector field $X \in \mathfrak{X}(M)$ is the unique vector field $\mathrm{curl}(X) \in \mathfrak{X}(M)$ satisfying $\iota_{\mathrm{curl}(X)} d\mathrm{vol} = d(X_\flat) \in \Omega^2(M)$, where $X_\flat := g(X, \cdot) \in \Omega^1(M)$.

(a) Show that in the special case of $M = \mathbb{R}^3$ with g chosen to be the standard Euclidean inner product,
$$\mathrm{curl}\begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} \partial_2 X^3 - \partial_3 X^2 \\ \partial_3 X^1 - \partial_1 X^3 \\ \partial_1 X^2 - \partial_2 X^1 \end{pmatrix} \in \mathfrak{X}(\mathbb{R}^3).$$

(b) The *gradient* $\nabla f \in \mathfrak{X}(M)$ of a function $f : M \to \mathbb{R}$ on a Riemannian manifold (M, g) is uniquely determined by the relation $g(\nabla f, \cdot) = df \in \Omega^1(M)$. Derive from $d \circ d = 0$ the following consequences for all smooth functions $f : M \to \mathbb{R}$ and vector fields $X \in \mathfrak{X}(M)$:
$$\mathrm{curl}(\nabla f) \equiv 0, \qquad \text{and} \qquad \mathrm{div}(\mathrm{curl}\, X) \equiv 0.$$

(c) Deduce from the Poincaré lemma the following results of classical vector analysis:

(i) Every vector field X with $\mathrm{curl}(X) \equiv 0$ is locally the gradient of a function.
(ii) Every vector field X with $\mathrm{div}(X) \equiv 0$ is locally the curl of another vector field.

Problem 2
Prove the following version of *integration by parts*: if M is a compact oriented n-manifold with boundary, $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$ with $k + \ell = n - 1$, then
$$\int_M d\alpha \wedge \beta = \int_{\partial M} \alpha \wedge \beta - (-1)^k \int_M \alpha \wedge d\beta.$$

Problem 3
Prove: For each $k \geqslant 0$, a k-form $\omega \in \Omega^k(M)$ is closed if and only for every compact oriented $(k+1)$-dimensional submanifold $L \subset M$ with boundary, $\int_{\partial L} \omega = 0$.

Problem 4
Prove: On S^1, a 1-form $\lambda \in \Omega^1(S^1)$ is exact if and only if $\int_{S^1} \lambda = 0$. What does this tell you about $H^1_{\mathrm{dR}}(S^1)$?
Hint: Try to construct a primitive $f : S^1 \to \mathbb{R}$ by integrating λ along paths.

Problem 5
Suppose $\mathcal{O}$ is an open subset of either $\mathbb{H}^n$ or $\mathbb{R}^n$. We call $\mathcal{O}$ a *star-shaped* domain if for every $p \in \mathcal{O}$, it also contains the points $tp \in \mathbb{R}^n$ for all $t \in [0, 1]$. It follows that $h(t, p) := tp$ defines a smooth homotopy $h : [0, 1] \times \mathcal{O} \to \mathcal{O}$ between the identity and the constant map whose value is the origin, making $\mathcal{O}$ smoothly contractible. Use this homotopy to produce an explicit formula for a linear operator $P : \Omega^k(\mathcal{O}) \to \Omega^{k-1}(\mathcal{O})$ for each $k \geqslant 1$ satisfying
$$\omega = P(d\omega) + d(P\omega)$$

for all $\omega \in \Omega^k(\mathcal{O})$. In particular, whenever ω is a closed k-form, $P\omega$ is a primitive of ω.
Hint: Start with the chain homotopy that we constructed in lecture for proving the homotopy invariance of de Rham cohomology. As a sanity check, the answer to this problem can be found at the end of Lecture 13 in the notes, but try to find it yourself first.

Problem 6
Show that the wedge product descends to an associative and graded-commutative product $\cup : H^k_{\mathrm{dR}}(M) \times H^\ell_{\mathrm{dR}}(M) \to H^{k+\ell}_{\mathrm{dR}}(M)$, defined by

$$[\alpha] \cup [\beta] := [\alpha \wedge \beta].$$

This is called the *cup product* on de Rham cohomology.
Remark: There is similarly a cup product on singular cohomology, to which this one is isomorphic via de Rham's theorem. But this one is easier to define, and is thus often used in practice as a surrogate for the singular cup product.

Problem 7
For this exercise, identify the n-torus $\mathbb{T}^n$ with the quotient $\mathbb{R}^n/\mathbb{Z}^n$ (recall from Problem Set 2 #1 that there is a natural diffeomorphism). For any sufficiently small open set $\widetilde{\mathcal{U}} \subset \mathbb{R}^n$, the usual Cartesian coordinates $x^1, \dots, x^n : \widetilde{\mathcal{U}} \to \mathbb{R}$ can be used to define a smooth chart $(\mathcal{U}, x)$ on $\mathbb{T}^n$ where

$$\mathcal{U} := \left\{ [p] \in \mathbb{T}^n \;\middle|\; p \in \widetilde{\mathcal{U}} \right\}, \qquad x([p]) := (x^1(p), \dots, x^n(p)) \text{ for } p \in \widetilde{\mathcal{U}}.$$

(a) Show that the coordinate differentials $dx^1, \dots, dx^n \in \Omega^1(\mathcal{U})$ arising from the chart $(\mathcal{U}, x)$ described above are independent of the choice of the set $\widetilde{\mathcal{U}} \subset \mathbb{R}^n$, i.e. the definitions of the coordinate differentials obtained from two different choices $\widetilde{\mathcal{U}}_1, \widetilde{\mathcal{U}}_2 \subset \mathbb{R}^n$ coincide on the region $\mathcal{U}_1 \cap \mathcal{U}_2 \subset \mathbb{T}^n$ where they overlap.

(b) As a consequence of part (a), the 1-forms $dx^1, \dots, dx^n \in \Omega^1(\mathbb{T}^n)$ are well-defined on the entire torus, and they are obviously locally exact and therefore closed, but they might not actually be exact since none of the coordinates $x^1, \dots, x^n$ admit smooth definitions globally on $\mathbb{T}^n$. Show in fact that for any constant vector $(a_1, \dots, a_n) \in \mathbb{R}^n \backslash \{0\}$, the 1-form

$$\lambda := a_i \, dx^i \in \Omega^1(\mathbb{T}^n)$$

is closed but not exact.
Hint: You only need to find one smooth map $\gamma : S^1 \to \mathbb{T}^n$ such that $\int_{S^1} \gamma^* \lambda \neq 0$.

(c) One can similarly produce closed k-forms $\omega \in \Omega^k(\mathbb{T}^n)$ for any $k \leqslant n$ by choosing constants $a_{i_1 \dots i_k} \in \mathbb{R}$ and writing

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(\mathbb{T}^n). \tag{1}$$

Show that for every nontrivial k-form of this type, one can find a cohomology class $[\alpha] \in H^{n-k}_{\mathrm{dR}}(\mathbb{T}^n)$ such that the cup product $[\omega] \cup [\alpha] \in H^n_{\mathrm{dR}}(\mathbb{T}^n)$ defined in Problem 4 is nontrivial, and deduce from this that ω is not exact.
Hint: Can you choose $\alpha \in \Omega^{n-k}(\mathbb{T}^n)$ so that $\omega \wedge \alpha$ is a volume form?

Remark: One can show that all cohomology classes in $H^k_{\mathrm{dR}}(\mathbb{T}^n)$ are representable by k-forms with constant coefficients as in (1)*, thus* $\dim H^k_{\mathrm{dR}}(\mathbb{T}^n) = \binom{n}{k}$.