Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, C. S. Hsueh

Differentialgeometrie I

WiSe 2024–25

Problem Set 8

To be discussed: 11.12.2024

Problem 1

Prove that for any closed symplectic manifold (M, ω), $H^2_{\mathrm{dR}}(M)$ is nontrivial.
Hint: What can you say about the n-fold cup product of $[\omega] \in H^2_{\mathrm{dR}}(M)$ with itself?

Problem 2

On a symplectic manifold (M, ω), the **Poisson bracket** of two smooth functions $F, G : M \to \mathbb{R}$ is the function $\{F, G\} : M \to \mathbb{R}$ defined by

$$\{F, G\} := \omega(X_F, X_G),$$

where X_F and X_G denote the Hamiltonian vector fields of F and G respectively.

(a) Write down an explicit formula for $\{F, G\}$ in terms of partial derivatives of F and G in local coordinates $(p^1, q^1, \ldots, p^n, q^n)$ such that $\omega = \sum_{j=1}^n dp^j \wedge dq^j$.
Remark: The resulting formula is the definition of $\{F, G\}$ found in most physics textbooks.

(b) Show that the following conditions are equivalent:

(i) The values of F are constant along all flow lines of X_G;
(ii) The values of G are constant along all flow lines of X_F;
(iii) $\{F, G\} \equiv 0$.

(c) Show that for any smooth function $H : M \to \mathbb{R}$, the values of H are constant along all flow lines of X_H.
Remark: In physics, the Hamiltonian function H is typically interpreted as the total energy of a mechanical system, in which case the result of this exercise is the basic principle known as conservation of energy.

Problem 3

Show that if X is a topological space with open subset $\mathcal{U} \subset X$ and a locally finite collection of continuous functions $\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in I}$ whose supports satisfy $\mathrm{supp}(f_\alpha) \subset \mathcal{U}$ for every $\alpha \in \mathcal{U}$, then $\sum_{\alpha \in I} f_\alpha$ also has support in $\mathcal{U}$.

Problem 4

Without mentioning Riemannian metrics, prove that a smooth n-manifold M admits a volume form $\omega \in \Omega^n(M)$ if and only if M is orientable.
Hint: If you were to take the existence of Riemannian metrics as given, then the existence of the volume form $\omega \in \Omega^n(M)$ would follow because every oriented Riemannian manifold has a canonical volume form. But do not use this. Try instead constructing ω directly, with the aid of a partition of unity.

Problem 5

Prove the following improvement on the theorem from lecture that every manifold M is paracompact: every open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of M admits a locally finite refinement $\{\mathcal{O}_\beta\}_{\beta \in J}$ in which each of the sets $\mathcal{O}_\beta$ is the domain of a chart.
Hint: The proof we worked through in lecture requires only one minor adjustment.