Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, C. S. Hsueh

Differentialgeometrie I
WiSe 2024–25

Problem Set 9

To be discussed: 18.12.2024

Notation: *As in the lectures,* $\mathbb{F}$ *denotes either* $\mathbb{R}$ *or* $\mathbb{C}$*, and all vector spaces, vector bundles and linear maps are over* $\mathbb{F}$ *unless otherwise specified. The dual of a vector space* V *is* $V^* := \mathrm{Hom}(V, \mathbb{F})$*, and* $\mathrm{Hom}(V, W)$ *denotes the space of linear maps* $V \to W$*.*

Problem 1

Suppose E is a smooth vector bundle (real of complex) of rank $m \geqslant 0$ over an n-manifold M. We saw in lecture that the total space of E admits an atlas of charts of the form

$$\phi_\alpha := (x_\alpha \times \mathbb{1}) \circ \Phi_\alpha : E|_{\mathcal{U}_\alpha} \to x_\alpha(\mathcal{U}_\alpha) \times \mathbb{F}^m \subset \mathbb{R}^n \times \mathbb{F}^m,$$

determined by open subsets $\mathcal{U}_\alpha \subset M$, charts $x_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n$ for M and local trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ for E.

(a) Write down the transition map $\phi_\beta \circ \phi_\alpha^{-1}$ relating any two charts of this form and convince yourself that it is smooth.

(b) Show that for the smooth structure on E determined by our atlas, the bundle projection $\pi : E \to M$, the inclusions of fibers $E_p \hookrightarrow E$ for each $p \in M$ and the zero-section

$$i : M \hookrightarrow E : p \mapsto 0 \in E_p$$

are all smooth maps. Show moreover that $\pi : E \to M$ is a submersion, while $E_p \hookrightarrow E$ and $i : M \hookrightarrow E$ are embeddings.

(c) By a theorem from the second lecture in this course, the atlas on E determines a natural topology, and before we're allowed to call E a "manifold", we must prove that this topology is metrizable. Prove this by constructing a Riemannian metric on E, using only the fact that M (but not necessarily E) is metrizable.
Hint: It would help to know that every open cover of E *admits a subordinate partition of unity, but you do not know this. You do know it however for* M*.*

Problem 2

For a smooth vector bundle E over M with local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$, every section $s : M \to E$ is determined on the subset $\mathcal{U}_\alpha \subset M$ by its so-called *local representation*, which is the unique function $s_\alpha : \mathcal{U}_\alpha \to \mathbb{F}^m$ such that

$$\Phi_\alpha(s(p)) = (p, s_\alpha(p)) \qquad \text{for all } p \in \mathcal{U}_\alpha.$$

Show that if $(\mathcal{U}_\alpha, \Phi_\alpha)$ and $(\mathcal{U}_\beta, \Phi_\beta)$ are two local trivializations of E and $s : M \to E$ is a section, then the local representations $s_\alpha : \mathcal{U}_\alpha \to \mathbb{F}^m$ and $s_\beta : \mathcal{U}_\beta \to \mathbb{F}^m$ are related to each other on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ in terms of the transition function $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})$ by

$$s_\beta(p) = g_{\beta\alpha}(p) s_\alpha(p) \qquad \text{for } p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

Problem 3

Suppose $E, F \to M$ are smooth vector bundles of ranks m and k respectively, and $\Psi : E \to F$ is a map whose restriction to E_p for each $p \in M$ is a linear map $\Psi_p : E_p \to F_p$.

(a) Show that for every pair of smooth local trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ and $\Psi_\beta : E|_{\mathcal{U}_\beta} \to \mathcal{U}_\beta \times \mathbb{F}^k$, there exists a unique function

$$\Psi_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{F}^{k\times m}$$

valued in the vector space of k-by-m matrices over $\mathbb{F}$ such that the map

$$(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{F}^m \xrightarrow{\Phi_\beta \circ \Psi \circ \Phi_\alpha^{-1}} (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{F}^k$$

takes the form $(p, v) \mapsto (p, \Psi_{\beta\alpha}(p)v)$.

(b) Show that $\Psi : E \to F$ is a smooth map if and only if for all choices of the two smooth local trivializations in part (a), the function $\Psi_{\beta\alpha}$ is smooth.

Problem 4

In lecture we considered a real line bundle ℓ over S^1, defined as follows: viewing S^1 as the unit circle in $\mathbb{C}$, define the set $\ell \subset S^1 \times \mathbb{R}^2$ as the union of the sets $\{e^{i\theta}\} \times \ell_{e^{i\theta}} \subset S^1 \times \mathbb{R}^2$ for all $\theta \in \mathbb{R}$, where the 1-dimensional subspace $\ell_{e^{i\theta}} \subset \mathbb{R}^2$ is given by

$$\ell_{e^{i\theta}} = \mathbb{R}\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix} \subset \mathbb{R}^2.$$

For any $\theta_0 \in \mathbb{R}$, we can set $p := e^{i\theta_0} \in S^1$ and define a local trivialization for ℓ over $S^1\backslash\{p\} \subset S^1$ by

$$\Phi : \ell|_{S^1\backslash\{p\}} \to (S^1\backslash\{p\}) \times \mathbb{R} : \left(e^{i\theta}, c\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix}\right) \mapsto (e^{i\theta}, c), \tag{1}$$

with θ assumed to vary in the interval $(\theta_0, \theta_0 + 2\pi)$. Prove:

(a) Any two local trivializations defined as in (1) with different choices of $\theta_0 \in \mathbb{R}$ are smoothly compatible.

(b) ℓ is a smooth subbundle of the trivial 2-plane bundle $S^1 \times \mathbb{R}^2$.

(c) There exists no continuous section of ℓ that is nowhere zero.

(d) ℓ is not globally trivial.

Problem 5

Here is a bit of preparatory multilinear algebra: define the **tensor product** $V \otimes W$ of two finite-dimensional vector spaces V, W to be the space of bilinear maps $V^* \times W^* \to \mathbb{F}$. The tensor product $v \otimes w \in V \otimes W$ of two vectors $v \in V$ and $w \in W$ can then be defined as the bilinear map $(v \otimes w)(\lambda, \mu) := \lambda(v)\mu(w)$. (In case you have seen a more general definition of the tensor product elsewhere, you can view this problem as an effort to convince you that that definition is equivalent to this one.)

(a) Verify that if $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are bases of V and W respectively, then the mn elements $v_i \otimes w_j \in V \otimes W$ for $i = 1, \ldots, m$ and $j, \ldots, n$ form a basis of $V \otimes W$.

(b) Show that for any vector space X, there is a canonical isomorphism between the space of linear maps $V \otimes W \to X$ and the space of *bilinear* maps $V \times W \to X$.

(c) Find a canonical isomorphism between $(V \otimes W) \otimes X$ and $V \otimes (W \otimes X)$ that identifies $(v \otimes w) \otimes x$ with $v \otimes (w \otimes x)$ for every $v \in V$, $w \in W$ and $x \in X$.
Hint: Identify both spaces with the space of all multilinear maps $V^* \times W^* \times X^* \to \mathbb{F}$. In the same manner, one can dispense with parentheses and identify any finite tensor product $V_1 \otimes \ldots \otimes V_k$ with the space of multilinear maps $V_1^* \times \ldots \times V_k^* \to \mathbb{F}$.