Take-Home Midterm

Due: Wednesday, 29.01.2025 (100pts total)

Instructions

The purpose of this assignment is three-fold:

- In the absence of regular problem sets for the next two weeks, it deals with current material from the lectures (especially Problems 2 and 3).
- It gives the instructors a chance to gauge your understanding more directly than usual, and give feedback.
- It provides an opportunity to improve your final grade in the course.

Concerning the first point: if you are in the habit of working through the problem sets regularly, then we strongly recommend that you work through and hand in this one as well, even if you know you cannot solve enough problems to have an impact on your grade. This pertains especially to Problems 2 and 3, since they involve material that has not been covered on any problem sets so far.

To receive feedback and/or credit, you must either hand in your written solutions before the beginning of the Übung on **Wednesday, January 29 at 9:15**, or scan and upload them to the moodle before that time. The solutions will be discussed in the Übung on that day.

You are free to use any resources at your disposal and to discuss the problems with your comrades, but **you must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

A score of 60 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus. The number of points assigned to each part of each problem is meant to be approximately proportional to its importance and/or difficulty, but this is not an exact science.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. if you need to choose a smooth cutoff function with particular properties and can justify its existence with a convincing picture instead of an explicit formula (use your own judgement). Unless stated otherwise, you are free to make use of all results that have appeared in the lecture notes or in problem sets, without reproving them. There is one **exception**: if a problem on this sheet has also been stated as an exercise in the lecture notes, then (obviously) proof is required. When using a result from a problem set or the lecture notes, say explicitly which one.

If you get stuck on one part of a problem, it may often still be possible to move on and do the next part. You are free to ask for clarification or hints via e-mail/moodle or in office hours or Übungen; of course we reserve the right not to answer such questions.

Problem 1 [30 pts]
Recall that on a symplectic manifold (M, ω), any smooth function $H : M \to \mathbb{R}$ gives rise to a so-called *Hamiltonian* vector field X_H, characterized uniquely by the condition $\omega(X_H, \cdot) = -dH$. The flow of X_H is symplectic, meaning it satisfies $(\varphi^t_{X_H})^*\omega = \omega$ for all t, or equivalently, $\mathcal{L}_{X_H}\omega \equiv 0$.

(a) [10 pts] Assume $H^1_{\mathrm{dR}}(M) = 0$ and $Y \in \mathfrak{X}(M)$ is a vector field with a globally-defined flow φ_t that is symplectic and leaves H invariant, meaning

$$\varphi_t^*\omega = \omega \quad \text{and} \quad H \circ \varphi_t \equiv H \qquad \text{for all } t. \tag{1}$$

Show that Y is then the Hamiltonian vector field for a function $F : M \to \mathbb{R}$ that is preserved under the flow of X_H, i.e. F is constant along all flow lines of X_H.

The result in part (a) is a simple case of a general principle known as Noether's theorem, which posits a bijective correspondence between symmetries of a mechanical system (such as the flow of the vector field Y) and quantities that are conserved under the motion of that system (e.g. the function F). A familiar example of a conserved quantity in mechanics is momentum, which corresponds to the invariance of the equations of motion under spatial translations. Here is another example.

Let $M = \mathbb{R}^4$ with coordinates (x, p_x, y, p_y) and the standard symplectic form

$$\omega_{\mathrm{std}} = dp_x \wedge dx + dp_y \wedge dy.$$

We can think of $\mathbb{R}^4$ as the "position-momentum space" (also called *phase space*) representing the motion of a single particle of mass $m > 0$ in a plane: its position is given by $\mathbf{q} := (x, y) \in \mathbb{R}^2$, and $\mathbf{p} := (p_x, p_y) \in \mathbb{R}^2$ are the corresponding "momentum variables". Given a "potential" function $V : \mathbb{R}^2 \to \mathbb{R}$, the total energy of the system is given by the function

$$H(\mathbf{q}, \mathbf{p}) = \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{q}).$$

Suppose now that the potential V is chosen to be *rotationally symmetric*, e.g. this is the case if $\mathbf{q}$ represents the position of the Earth moving around the sun (with the latter positioned at the origin). To express this condition succinctly, one can transform to polar coordinates (r, θ) on $\mathbb{R}^2$, related to the (x, y)-coordinates as usual by $x = r\cos\theta$ and $y = r\sin\theta$. The condition imposed on V is then $\partial_\theta V \equiv 0$.

(b) [10 pts] Regarding r and θ as real-valued functions on (a suitable subdomain of) $\mathbb{R}^4$ that depend on the coordinates x and y but not on p_x and p_y, define two additional functions on the same domain by

$$p_r := \frac{x}{r}p_x + \frac{y}{r}p_y, \qquad p_\theta := -yp_x + xp_y.$$

Show that $(r, p_r, \theta, p_\theta)$ is then a Darboux chart with respect to the symplectic form ω_{std}, meaning $\omega_{\mathrm{std}} = dp_r \wedge dr + dp_\theta \wedge d\theta$.
Hint: A direct computation of ω_{std} in the new coordinates would suffice, but this computation is a bit long. You could make your life easier by observing that $\omega_{\mathrm{std}} = d\lambda_{\mathrm{std}}$ for $\lambda_{\mathrm{std}} := p_x\,dx + p_y\,dy$, and then computing λ_{std} in the new coordinates.

(c) [10 pts] Write down H as a function of $(r, p_r, \theta, p_\theta)$ and show that the family of diffeomorphisms defined in these coordinates by $\varphi_t(r, p_r, \theta, p_\theta) := (r, p_r, \theta + t, p_\theta)$ satisfies (1). Derive a formula for the corresponding conserved quantity F as promised by part (a). *(It is called the "angular momentum," for reasons that should now appear somewhat natural.)*

Problem 2 [42 pts]
Throughout this problem, suppose M is a smooth manifold with a symmetric connection ∇ on its tangent bundle, and the associated tensor bundles $T^k_\ell M \to M$ are equipped with the connections naturally determined by ∇. Assuming $X, Y, Z \in \mathfrak{X}(M)$, prove:

(a) [8 pts] For any $\lambda \in \Omega^1(M)$, $d\lambda(X,Y) = (\nabla_X \lambda)(Y) - (\nabla_Y \lambda)(X)$.

(b) [8 pts] For any $\lambda \in \Omega^1(M)$, $(\mathcal{L}_X \lambda)(Y) = (\nabla_X \lambda)(Y) + \lambda(\nabla_Y X)$.

(c) [10 pts] For any $S \in \Gamma(T^0_2 M)$, $(\mathcal{L}_X S)(Y,Z) = (\nabla_X S)(Y,Z) + S(\nabla_Y X, Z) + S(Y, \nabla_Z X)$. *Hint: It suffices (why?) to verify this for tensor fields of the form $\lambda \otimes \mu \in \Gamma(T^0_2 M)$ with $\lambda, \mu \in \Omega^1(M)$. How does the operator $\mathcal{L}_X$ to behave under tensor products?*

(d) [10 pts] Assume ∇ is the Levi-Cività connection for a pseudo-Riemannian metric $g = \langle\ ,\ \rangle$ on M. For each $p \in M$, g determines the so-called *musical isomorphisms*

$$T_pM \xrightarrow{\flat} T_p^*M : X \mapsto X_\flat := \langle X, \cdot \rangle, \qquad \text{and} \qquad T_p^*M \xrightarrow{\sharp := \flat^{-1}} T_pM : \lambda \mapsto \lambda^\sharp,$$

which similarly associate to each vector field $X \in \mathfrak{X}(M)$ a 1-form $X_\flat \in \Omega^1(M)$. Show that for $X \in \mathfrak{X}(M)$, the type $(0,2)$ tensor field $\nabla(X_\flat) \in \Gamma(T^0_2 M)$ is *antisymmetric* (i.e. it is a differential 2-form) if and only if X satisfies the relation

$$\langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle \equiv 0 \qquad \text{for all } Y, Z \in \mathfrak{X}(M). \tag{2}$$

Equation (2) is known as the *Killing equation*.

(e) [6 pts] Assuming M is compact, what can you say about the flow of a vector field $X \in \mathfrak{X}(M)$ if X satisfies the Killing equation (2)?

Problem 3 [28 pts]
One of the standard examples of "non-Euclidean" geometry is a Riemannian manifold known as the *Poincaré half-plane* $(\mathbb{H}, h)$. It is the smooth 2-manifold $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the Riemannian metric $h \in \Gamma(T^0_2 \mathbb{H})$ defined by

$$h_{(x,y)}(X,Y) := \frac{1}{y^2} \langle X, Y \rangle_E, \qquad \text{for } X, Y \in T_{(x,y)}\mathbb{H} = \mathbb{R}^2,$$

where $\langle\ ,\ \rangle_E$ denotes the standard Euclidean inner product on $\mathbb{R}^2$.

(a) [10 pts] Show that a smooth path $\gamma(t) = (x(t), y(t)) \in \mathbb{H}$ is a geodesic on $(\mathbb{H}, h)$ if and only if it satisfies the following second-order system of ordinary differential equations:

$$\begin{aligned} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} &= 0 \\ \ddot{y} + \frac{1}{y}\left(\dot{x}^2 - \dot{y}^2\right) &= 0. \end{aligned} \tag{3}$$

Advice: I'm not sure whether there is a cleverer way to do this, but it is definitely doable by directly computing the Christoffel symbols. Try not to compute any more than you actually have to, e.g. remember that since the connection is symmetric, some of the Christoffel symbols determine some of the others.

(b) [6 pts] Show that for any constants $x_0 \in \mathbb{R}$ and $r > 0$, the equations (3) admit solutions of the form

$$(x(t), y(t)) = (x_0, y(t))$$

for some function $y(t) > 0$, as well as

$$(x(t), y(t)) = (x_0 + r\cos\theta(t), r\sin\theta(t)) \,.$$

for some function $\theta(t) \in (0, \pi)$.

(c) [4 pts] Do you think that *all* of the geodesics on $(\mathbb{H}, h)$ have the form described in part (b)? Answer with a brief heuristic argument, preferably based on a picture.

(d) [8 pts] Prove that $\exp(Y)$ is well defined for all tangent vectors $Y \in T\mathbb{H}$ that point upward or downward, i.e. that are proportional to ∂_y.
Hint: What is the length of a geodesic of the form $\gamma(t) = (x_0, y(t))$*?*