PROBLEM SET 12

There is essentially just one problem this week, and it is to compute the cohomology ring $H^*(\mathbb{T}^n; R)$ for every $n \geqslant 1$, with coefficients in any commutative ring R with unit. The idea is to derive the cup product from information about the homological cross product

$$H_k(\mathbb{T}^m; R) \otimes H_\ell(\mathbb{T}^n; R) \xrightarrow{\times} H_{k+\ell}(\mathbb{T}^{m+n}; R),$$

using the relations

$$\langle [\varphi] \times [\psi], [a] \times [b] \rangle = (-1)^{|a| \cdot |\psi|} \langle [\varphi], [a] \rangle \langle [\psi], [b] \rangle \tag{1}$$

and

$$[\varphi] \times [\psi] = \pi_X^*[\varphi] \cup \pi_Y^*[\psi] \tag{2}$$

for $[\varphi] \in H^*(X)$, $[\psi] \in H^*(Y)$, $[a] \in H_*(X)$ and $[b] \in H_*(Y)$, with π_X and π_Y denoting the projections of $X \times Y$ to X and Y respectively.

The homology of $\mathbb{T}^n$ is fairly easy to compute because $\mathbb{T}^n = S^1 \times \ldots \times S^1$ has a natural structure as a product cell complex (cf. Exercise 46.1 in the lecture notes). Without mentioning cell complexes, we can also use an inductive argument based on the Künneth formula. Indeed, the case $n = 1$ is trivial since $\mathbb{T}^1 = S^1$, so in particular, $H_*(S^1; \mathbb{Z})$ is a finitely generated free abelian group. Let's call its canonical generators

$$[*] \in H_0(S^1; \mathbb{Z}), \qquad [S^1] \in H_1(S^1; \mathbb{Z}),$$

i.e. $[*]$ is the homology class represented by any singular 0-simplex $\Delta^0 \to S^1$, and $[S^1]$ is the class represented by the identity map $S^1 \to S^1$ under the isomorphism $H_1(S^1; \mathbb{Z}) \cong \pi_1(S^1)$.

(a) Derive from the Künneth formula an isomorphism

$$H_m(\mathbb{T}^n; \mathbb{Z}) \cong H_{m-1}(\mathbb{T}^{n-1}; \mathbb{Z}) \oplus H_m(\mathbb{T}^{n-1}; \mathbb{Z})$$

for every $n \geqslant 2$ and $m \in \mathbb{Z}$. Deduce that $H_m(\mathbb{T}^n; \mathbb{Z})$ is always a finitely-generated abelian group, whose rank is an entry in Pascal's triangle,

$$\operatorname{rank} H_m(\mathbb{T}^n; \mathbb{Z}) = \binom{n}{m}.$$

Hint: Since you already know that $H_*(S^1; \mathbb{Z})$ is finitely generated and free, you can prove by induction on $n \in \mathbb{N}$ that the same is true for $H_*(\mathbb{T}^n; \mathbb{Z})$; this should remove any need to worry about Tor *terms.*

(b) For each $m \in \mathbb{N}$ and each choice of integers $1 \leqslant j_1 < \ldots < j_m \leqslant n$, define the homology class

$$e_{j_1, \ldots, j_m} := A_1 \times \ldots \times A_n \in H_m(\mathbb{T}^n; \mathbb{Z})$$

by setting $A_{j_i} := [S^1]$ for each $i = 1, \ldots, m$ and $A_j := [*]$ for all other $j \in \{1, \ldots, n\}$. Deduce from the Künneth formula that the set of all such elements forms a basis of $H_*(\mathbb{T}^n; \mathbb{Z})$.

It will be useful to have an alternative description of the degree 1 generators $e_j \in H_1(\mathbb{T}^n; \mathbb{Z})$ that appear in part (b). Pick a base point $t_0 \in S^1$ and consider the embedding

$$i_j : S^1 \hookrightarrow \mathbb{T}^n : x \mapsto (\underbrace{t_0 \times \ldots \times t_0}_{j-1}, x, \underbrace{t_0 \times \ldots \times t_0}_{n-j}). \tag{3}$$

Note that different choices of the base point $t_0 \in S^1$ give homotopic maps $i_j : S^1 \to \mathbb{T}^n$, thus the induced map $(i_j)_* : H_*(S^1; \mathbb{Z}) \to H_*(\mathbb{T}^n; \mathbb{Z})$ is independent of this choice.

(c) Show that for each $j = 1, \dots, n$, $(i_j)_*[S^1] = e_j$.
Hint: For the case $j = n$, you can identify S^1 in the obvious way with $\{*\} \times S^1$ and then write $i_n : S^1 \hookrightarrow \mathbb{T}^n$ as
$$i_n = \iota \times \mathrm{Id} : \{*\} \times S^1 \hookrightarrow \mathbb{T}^{n-1} \times S^1,$$
with $\iota : \{*\} \to \mathbb{T}^{n-1}$ denoting the inclusion of the point $(t_0, \dots, t_0)$. The naturality of the cross product then gives a commutative diagram
$$\begin{array}{ccc} H_0(\{*\};\mathbb{Z}) \otimes H_1(S^1;\mathbb{Z}) & \xrightarrow{\times} & H_1(S^1;\mathbb{Z}) \\ \downarrow{\scriptstyle \iota_* \otimes \mathbb{1}} & & \downarrow{\scriptstyle (i_n)_*} \\ H_0(\mathbb{T}^{n-1};\mathbb{Z}) \otimes H_1(S^1;\mathbb{Z}) & \xrightarrow{\times} & H_1(\mathbb{T}^n;\mathbb{Z}). \end{array}$$
Use the fact that $[*] \times [S^1] = [S^1]$ for the canonical generator $[*] \in H_0(\{*\};\mathbb{Z})$, after identifying $\{*\} \times S^1 = S^1$, while $\iota_* : H_0(\{*\};\mathbb{Z}) \to H_0(\mathbb{T}^{n-1};\mathbb{Z})$ is an isomorphism relating the canonical generators.

(d) Use the universal coefficient theorem to upgrade the computation of $H_*(\mathbb{T}^n;\mathbb{Z})$ above to a computation of $H_*(\mathbb{T}^n;G)$ for arbitrary coefficient groups G. Deduce in particular that for any commutative ring R with unit, $H_*(\mathbb{T}^n;R)$ has the structure of a finitely-generated free R-module, with a basis in bijective correspondence with the basis of $H_*(\mathbb{T}^n;\mathbb{Z})$ described above.
Remark: There is no need to assume that R is a principal ideal domain here, as we are only using the universal coefficient theorem and Künneth formula for $\mathbb{Z}$-modules.

(e) Show that for each $m \in \mathbb{Z}$, the natural map
$$H^m(\mathbb{T}^n;R) \to \mathrm{Hom}(H_m(\mathbb{T}^n;R),R) : [\varphi] \mapsto \langle [\varphi], \cdot \rangle$$
is an R-module isomorphism, implying $H^m(\mathbb{T}^n;R) \cong R^{\binom{n}{m}}$.
Hint: Again, you do not need to assume here that R is a principal ideal domain. Just regard R at first as an abelian group, prove that the natural map from $H^m(\mathbb{T}^n;R)$ to the abelian group of group homomorphisms $H_m(\mathbb{T}^n;\mathbb{Z}) \to R$ is a group isomorphism, and then use what you know about the algebraic structure of $H_m(\mathbb{T}^n;R)$.

Henceforth, we fix R as the coefficient ring for both $H_*(\mathbb{T}^n)$ and $H^*(\mathbb{T}^n)$ and omit it from the notation wherever possible, regarding these groups as R-modules. We can write down a canonical basis for $H^*(\mathbb{T}^n)$ as follows. For $n = 1$, define
$$\lambda \in H^1(S^1)$$
to be the unique cohomology class such that
$$\langle \lambda, [S^1] \rangle = 1 \in R.$$
Now for each choice of integers $1 \leqslant j_1 < \dots < j_m \leqslant n$, define
$$\lambda_{j_1,\dots,j_m} := \alpha_1 \times \dots \times \alpha_n \in H^m(\mathbb{T}^n),$$
where we choose $\alpha_{j_i} := \lambda$ for each $i = 1, \dots, m$ and $\alpha_j = 1 \in H^0(S^1)$ for all other $j \in \{1, \dots, n\}$. By (1), we have
$$\begin{aligned} \langle \lambda_{j_1,\dots,j_m}, e_{k_1,\dots,k_m} \rangle &= \langle \alpha_1 \times \dots \times \alpha_n, A_1 \times \dots \times A_n \rangle = \pm \langle \alpha_1, A_1 \rangle \dots \langle \alpha_n, A_n \rangle \\ &= \begin{cases} \pm 1 & \text{if } j_i = k_i \text{ for all } i = 1, \dots, m, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$
proving that the collection of classes $\lambda_{j_1,\dots,j_m}$ for all choices $1 \leqslant j_1 < \dots < j_m \leqslant n$ is a basis for $H^*(\mathbb{T}^n)$ as a free R-module.

To describe $H^*(\mathbb{T}^n)$ as a ring, we now need to compute each product of the form $\lambda_{j_1,\dots,j_m} \cup \lambda_{k_1,\dots,k_q} \in H^{m+q}(\mathbb{T}^n)$. We start with an observation about the 1-dimensional classes $\lambda_j \in H^1(\mathbb{T}^n)$. Consider for each $j = 1, \dots, n$ the projection map

$$\pi_j : \mathbb{T}^n \to S^1 : (x_1, \dots, x_n) \mapsto x_j,$$

which is related to the inclusions $i_j : S^1 \hookrightarrow \mathbb{T}^n$ defined in (3) above by

$$\pi_j \circ i_k = \begin{cases} \mathrm{Id} : S^1 \to S^1 & \text{if } j = k, \\ \text{constant} & \text{if } j \neq k. \end{cases}$$

(f) Show that $\pi_j^* \lambda = \lambda_j$ for each $j = 1, \dots, n$.
Hint: Evaluate both $\pi_j^* \lambda$ and λ_j on the generators $e_i \in H_1(\mathbb{T}^n)$.

(g) Use (2) to prove that for any $m \in \mathbb{N}$ and integers $1 \leqslant j_1 < \dots < j_m \leqslant n$,

$$\lambda_{j_1} \cup \dots \cup \lambda_{j_m} = \lambda_{j_1,\dots,j_m}.$$

Conclude that the ring $H^*(\mathbb{T}^n; R)$ is isomorphic to the exterior algebra $\Lambda_R[\lambda_1, \dots, \lambda_n]$ over R on n generators of degree 1.