PROBLEM SET 13

1. The **smash product** $X \wedge Y$ of two pointed spaces (X, x_0) and (Y, y_0) is defined by

$$X \wedge Y := (X \times Y) \Big/ \big((\{x_0\} \times Y) \cup (X \times \{y_0\})\big) \cong (X \times Y)/(X \vee Y).$$

Any two base-point preserving maps $f : (X, x_0) \to (X', x_0')$ and $g : (Y, y_0) \to (Y', y_0')$ then define a product map $f \times g : X \times Y \to X' \times Y'$ which descends continuously to the quotient as a map

$$f \wedge g : X \wedge Y \to X' \wedge Y'.$$

A popular example of the smash product is furnished by spheres: if we identify S^n with $\mathbb{D}^n/\partial\mathbb{D}^n$, regarding the equivalence class of the boundary as the base point, and then choose a homeomorphism to identify the disk $\mathbb{D}^n$ with the cube I^n, then the obvious identification of $I^k \times I^\ell$ with $I^{k+\ell}$ descends to a homeomorphism

$$S^k \wedge S^\ell \cong (I^k/\partial I^k) \wedge (I^\ell/\partial I^\ell) \cong I^{k+\ell}/\partial I^{k+\ell} \cong S^{k+\ell} \qquad \text{for all} \qquad k, \ell \geqslant 0.$$

If X and Y are CW-complexes and we choose the base points to be 0-cells in these complexes, then the two subspaces $X \times \{y_0\}$ and $\{x_0\} \times Y$ in $X \times Y$ form an excisive couple, so that the relative cross product and Künneth formula are valid for the pairs $(X, \{x_0\})$ and $(Y, \{y_0\})$. Since $(X, x_0) \times (Y, y_0) = (X \times Y, X \vee Y)$, the Künneth formula then takes the form

$$0 \to \bigoplus_{k+\ell=n} H_k(X, \{x_0\}) \otimes H_\ell(Y, \{y_0\}) \xrightarrow{\times} H_n(X \times Y, X \vee Y) \longrightarrow \bigoplus_{k+\ell=n-1} \operatorname{Tor}(H_k(X, \{x_0\}), H_\ell(Y, \{y_0\})) \to 0,$$

which can be rewritten as

$$0 \to \bigoplus_{k+\ell=n} \widetilde{H}_k(X) \otimes \widetilde{H}_\ell(Y) \xrightarrow{\times} \widetilde{H}_n(X \wedge Y) \longrightarrow \bigoplus_{k+\ell=n-1} \operatorname{Tor}(\widetilde{H}_k(X), \widetilde{H}_\ell(Y)) \to 0$$

since CW-pairs (X, A) are also "good pairs", so that $H_*(X, A) \cong \widetilde{H}_*(X/A)$.

   (a) Show that for the cross product on reduced homology as described above and the identification of $S^k \wedge S^\ell$ with $S^{k+\ell}$, if $[S^k] \in \widetilde{H}_k(S^k; \mathbb{Z})$ and $[S^\ell] \in \widetilde{H}_\ell(S^\ell; \mathbb{Z})$ are generators, then $[S^k] \times [S^\ell] \in \widetilde{H}_{k+\ell}(S^{k+\ell}; \mathbb{Z})$ is also a generator.

   (b) Use the naturality of the Künneth formula to prove that for any two base-point preserving maps $f : S^k \to S^k$ and $g : S^\ell \to S^\ell$, $\deg(f \wedge g) = \deg(f) \cdot \deg(g)$.

   (c) Find an alternative proof of the formula in part (b) by counting preimages of points. Feel free to use any facts about smooth maps and perturbations that may seem convenient.

   (d) Using the definition of cellular chain maps and the cellular cross product, prove that the cellular cross product is natural, i.e. if $f : X \to X'$ and $g : Y \to Y'$ are cellular maps, then the diagram

$$\begin{array}{ccc} C_*^{\mathrm{CW}}(X) \otimes C_*^{\mathrm{CW}}(Y) & \xrightarrow{\times} & C_*^{\mathrm{CW}}(X \times Y) \\ \downarrow{\scriptstyle f_* \otimes g_*} & & \downarrow{\scriptstyle (f \times g)_*} \\ C_*^{\mathrm{CW}}(X') \otimes C_*^{\mathrm{CW}}(Y') & \xrightarrow{\times} & C_*^{\mathrm{CW}}(X' \times Y') \end{array}$$

   commutes.

2. Prove that if M is a non-orientable connected topological manifold, then $\pi_1(M)$ contains a subgroup of index 2. (In particular, this implies that every simply connected manifold is orientable.)

3. Remind yourself what was proved in Exercise 45.13 in the lecture notes, and how, and then use it to deduce that for any compact orientable topological n-manifold, $H_{n-1}(M;\mathbb{Z})$ is free and $H^n(M;\mathbb{Z}) \cong H_n(M;\mathbb{Z})$.

4. Fix a coefficient ring R. If M is a compact n-manifold with boundary, an **R-orientation** of M is defined to be an R-orientation of its interior, i.e. a section $s \in \Gamma(\Theta^R|_{\mathring{M}})$ such that $s(x)$ generates the R-module $\Theta^R_x = H_n(M|x) \cong R$ for every $x \in \mathring{M} := M \backslash \partial M$. The **relative fundamental class** of M is then the unique class $[M] \in H_n(M, \partial M)$ such that the map induced by the inclusion $(M, \partial M) \hookrightarrow (M, M\backslash\{x\})$ sends $[M]$ to $s(x)$ for every $x \in \mathring{M}$.

   (a) Show that if M and ∂M are both connected and ∂M is nonempty, then ∂M is also R-orientable, and the connecting homomorphism $\partial_* : H_n(M, \partial M) \to H_{n-1}(\partial M)$ in the long exact sequence of $(M, \partial M)$ is an isomorphism sending $[M]$ to the fundamental class $[\partial M]$ of ∂M (for a suitable choice of orientation of ∂M).
   Hint: Focus on the case $R = \mathbb{Z}$. It is easy to prove that ∂_* is injective; show that if it were not surjective, then $H_{n-1}(M;\mathbb{Z})$ would have torsion, contradicting the result of Problem 3.

   (b) Generalize the result of part (a) to prove $\partial_*[M] = [\partial M]$ without assuming ∂M is connected.
   Hint: For any connected component $N \subset \partial M$, consider the exact sequence of the triple $(M, \partial M, \partial M\backslash N)$ and notice that $H_{n-1}(\partial M, \partial M\backslash N) \cong H_{n-1}(N)$ by excision.

   (c) Conclude that for any compact manifold M with boundary and an R-orientation, the map $H_{n-1}(\partial M) \to H_{n-1}(M)$ induced by the inclusion $\partial M \hookrightarrow M$ sends $[\partial M]$ to 0. In other words, "the boundary of a compact oriented n-manifold M represents the trivial homology class in $H_{n-1}(M)$."
   Remark: We discussed a similar result in the setting of triangulable manifolds early in the semester, but here we are not assuming that any of our manifolds admit triangulations. One can use this to define natural transformations from bordism theories to singular homology, e.g. for unoriented bordism theory, the existence of the fundamental class $[M] \in H_n(M;\mathbb{Z}_2)$ for any closed n-manifold gives rise to homomorphisms

$$\Omega_n^{\mathrm{O}}(X) \to H_n(X;\mathbb{Z}_2) : [(M, \varphi)] \mapsto \varphi_*[M].$$