PROBLEM SET 14

1. Early in this semester, we associated to any compact triangulated n-manifold $(M, \partial M) \cong (|K|, |L|)$ with underlying simplicial pair (K, L) a distinguished simplicial homology class $[M] \in H_n^{\Delta}(K, L; \mathbb{Z}_2)$, represented by an n-cycle formed by summing all the n-simplices in the triangulation. The image of this class under the natural isomorphism $H_n^{\Delta}(K, L; \mathbb{Z}_2) \to H_n(M, \partial M; \mathbb{Z}_2)$ is then a distinguished singular homology class $[M] \in H_n(M, \partial M; \mathbb{Z}_2)$.
   (a) Show that $[M] \in H_n(M, \partial M; \mathbb{Z}_2)$ is the fundamental class of M associated to its unique $\mathbb{Z}_2$-orientation.
   (b) Prove the analogous statement about a distinguished class $[M] \in H_n(M, \partial M; \mathbb{Z})$ in the case where the triangulation is oriented.

2. The goal of this problem is to prove that the product $M \times N$ of two R-oriented manifolds inherits a natural R-orientation, and in the compact case, the associated fundamental class $[M \times N]$ is given by the cross product $[M] \times [N]$. Note that if M and N are topological manifolds of dimensions m and n respectively with boundary, then $M \times N$ is a topological $(m+n)$-manifold with boundary

$$\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N),$$

   thus in terms of the product we defined for pairs of spaces,

$$(M, \partial M) \times (N, \partial N) = (M \times N, \partial(M \times N)).$$

   For a fixed choice of coefficient ring R, we denote the corresponding orientation bundle of an arbitrary n-manifold M by

$$\Theta^M \xrightarrow{p} \mathring{M}, \qquad \Theta^M_x = H_n(M \,|\, x; R) \cong R.$$

   (a) Show that for any coefficient ring R, the map

$$H_m(\mathbb{D}^m, \partial \mathbb{D}^m; R) \otimes H_n(\mathbb{D}^n, \partial \mathbb{D}^n; R) \xrightarrow{\times} H_{m+n}(\mathbb{D}^m \times \mathbb{D}^n, \partial(\mathbb{D}^m \times \mathbb{D}^n); R)$$

   is an isomorphism.
   Remark: The Künneth formula offers one convenient approach to this, but only if R is a principal ideal domain. Try to do without that assumption.

   (b) Given an m-manifold M and an n-manifold N with interior points $x \in \mathring{M}$ and $y \in \mathring{N}$, we have

$$(M, M \backslash \{x\}) \times (N, N \backslash \{y\}) = (M \times N, (M \times N) \backslash \{(x, y)\}),$$

   so that the relative cross product defines a map

$$\Theta^M_x \otimes \Theta^N_y \xrightarrow{\times} \Theta^{M \times N}_{(x,y)}.$$

   Show that this map is an isomorphism, and that it gives rise to a homomorphism

$$\Gamma(\Theta^M) \otimes \Gamma(\Theta^N) \xrightarrow{\times} \Gamma(\Theta^{M \times N}) : s \otimes t \mapsto s \times t$$

   given by $(s \times t)(x, y) = s(x) \times t(y)$. Conclude that if $s \in \Gamma(\Theta^M)$ and $t \in \Gamma(\Theta^N)$ are R-orientations, then so is $s \times t \in \Gamma(\Theta^{M \times N})$.

   (c) Deduce via the naturality of the cross product with respect to maps of the form $(M, \partial M) \to (M, M \backslash \{x\})$ and $(N, \partial N) \to (N, N \backslash \{y\})$ that if M and N are compact manifolds with R-orientations and $M \times N$ is equipped with the product R-orientation arising from part (b), then the corresponding fundamental classes $[M] \in H_m(M, \partial M; R)$, $[N] \in H_n(N, \partial N; R)$ and $[M \times N] \in H_{m+n}(M \times N, \partial(M \times N); R)$ are related by

$$[M] \times [N] = [M \times N].$$

3. Use the nonsingularity of the intersection form to establish the following isomorphisms of $\mathbb{Z}$-graded rings.

   (a) $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 2$
   (b) $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 1$

   Hint for both: You need to show in both cases that if $\alpha \in H^k(M;R) \cong R$ and $\beta \in H^\ell(M;R) \cong R$ are generators with $k + \ell \leqslant \dim M$, then $\alpha \cup \beta$ is a generator of $H^{k+\ell}(M;R) \cong R$. Start with the case $k + \ell = \dim M$, and then deduce the general case from this using the fact that for each $m = 0,\dots,n$, there are natural inclusions $\mathbb{RP}^m \hookrightarrow \mathbb{RP}^n$ and $\mathbb{CP}^m \hookrightarrow \mathbb{CP}^n$ which are also cellular maps. (The induced homomorphisms on cohomology should be easy to compute.)

   Now use the obvious (cellular) inclusions $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^\infty$ and $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$ to compute:

   (c) $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]$ with $|\alpha| = 2$
   (d) $H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$ with $|\alpha| = 1$

4. A closed and connected 3-manifold M is called a **rational homology sphere** if $H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$. Prove that this condition holds if and only if M is orientable and $H_1(M;\mathbb{Z})$ is torsion.

5. In lecture we defined the **compactly supported cohomology** $H_c^*(X)$ of a space X via the direct limit

$$H_c^k(X) := \varinjlim \left\{H^k(X \,|\, K)\right\}_K ,$$

   where $H^k(X\,|\,K)$ is an abbreviation for $H^k(X, X\backslash K)$, and K ranges over the set of all compact subsets of X. These subsets are ordered by inclusion $K \subset K' \subset X$ and form a direct system via the maps $H^k(X\,|\,K) \to H^k(X\,|\,K')$ induced by inclusions $(X, X\backslash K') \hookrightarrow (X, X\backslash K)$.

   (a) Letting G denote the (arbitrarily chosen) coefficient group, construct a canonical isomorphism between $H_c^*(X)$ and the homology of the subcomplex $C_c^*(X) \subset C^*(X)$ consisting of every cochain $\varphi : C_k(X) \to G$ for which there exists a compact subset $K \subset X$ with $\varphi|_{C_k(X\backslash K)} = 0$. (Note that K may depend on φ).
   (b) Recall that a continuous map $f : X \to Y$ is called **proper**[1] if for every compact set $K \subset Y$, $f^{-1}(K) \subset X$ is also compact. Show that proper maps $f : X \to Y$ induce homomorphisms $f^* : H_c^*(Y) \to H_c^*(X)$, making H_c^* into a contravariant functor on the category of topological spaces with morphisms defined as proper maps.
   (c) Deduce from part (b) that H_c^* is a topological invariant, i.e. $H_c^*(X)$ and $H_c^*(Y)$ are isomorphic whenever X and Y are homeomorphic. Give an example showing that this need not be true if X and Y are only homotopy equivalent.
   (d) In contrast to part (b), show that H_c^* does *not* define a functor on the usual category of topological spaces with morphisms defined to be continuous (but not necessarily proper) maps.
   Hint: Think about maps between $\mathbb{R}^n$ and the one-point space.
   (e) We say that two proper maps $f, g : X \to Y$ are **properly homotopic** if there exists a homotopy $h : I \times X \to Y$ between them that is also a proper map. Show that under this assumption, the induced maps $f^*, g^* : H_c^*(Y) \to H_c^*(X)$ in part (b) are identical. In other words, H_c^* defines a contravariant functor on the category whose objects are topological spaces and whose morphisms are proper homotopy classes of proper maps.
   Hint: If you express H_c^* as the homology of the cochain complex in part (a), then your main task is to show that the usual dualized chain homotopy $h^* : C^*(Y) \to C^{*-1}(X)$ induced by h sends $C_c^*(Y)$ to $C_c^{*-1}(X)$. Alternatively, it should also be possible to work with the definition of H_c^* via direct limits and use the universal property to characterize the maps $f^*, g^* : H_c^*(Y) \to H_c^*(X)$. In both approaches, you may find it helpful to know that every compact subset of $I \times X$ is contained in $I \times K'$ for some compact $K' \subset X$.

[1] This definition of properness is standard in differential geometry, though for certain purposes, it is sometimes considered an inadequate definition if considering spaces that are not assumed second countable and Hausdorff (the general definition of properness is then a slightly stronger condition). As far as I can tell, it's still an adequate definition for the present exercise.