PROBLEM SET 15

1. For two closed, connected and oriented manifolds M, N of dimension $n \in \mathbb{N}$, the **degree** $\deg(f) \in \mathbb{Z}$ of a map $f : M \to N$ is defined to be the unique integer d such that

$$H_n(M;\mathbb{Z}) \xrightarrow{f_*} H_n(N;\mathbb{Z}) : [M] \mapsto d[N].$$

Recall that for any field $\mathbb{K}$, the intersection form

$$H^k(M;\mathbb{K}) \otimes H^{n-k}(M;\mathbb{K}) \xrightarrow{Q} \mathbb{K} : \alpha \otimes \beta \mapsto Q(\alpha,\beta) := \langle \alpha \cup \beta, [M] \rangle$$

is nonsingular as a corollary of Poincaré duality; here $[M] \in H_n(M;\mathbb{K})$ is the fundamental class resulting from the fact that every orientation of M determines a $\mathbb{K}$-orientation via the universal coefficient theorem. It follows in particular that for every $\alpha \neq 0 \in H^k(M;\mathbb{K})$, one has $Q(\alpha,\beta) = 1$ for some $\beta \in H^{n-k}(M;\mathbb{K})$. Use this to prove that whenever a map $f : M \to N$ has $\deg(f) = d \neq 0$ and $\mathbb{K}$ is a field whose characteristic does not divide d, the induced maps $f^* : H^k(N;\mathbb{K}) \to H^k(M;\mathbb{K})$ and $f_* : H_k(M;\mathbb{K}) \to H_k(N;\mathbb{K})$ are injective and surjective respectively for every k.

2. In addition to the assumptions of Problem 1, suppose the manifolds M and N are smooth and $f : M \to N$ is a smooth map. Use the intersection product $f_*[M] \cdot [*]$ with the homology class of a point $[*] \in H_0(N;\mathbb{Z})$ to prove that under a suitable technical assumption on a point $y \in N$, $\deg(f)$ is a signed count of points in $f^{-1}(y) \subset M$. What exactly is the technical assumption you need?
Remark: Earlier in the course, we proved a generalization of this statement that applied to continuous maps between topological n-manifolds and only required the condition that $f^{-1}(y) \subset M$ is a finite set. You'll find however that the proof in this exercise is much quicker.

3. Throughout this problem, M is a closed, connected and oriented smooth manifold of dimension $n \in \mathbb{N}$. We consider intersection products of homology classes with complementary degree and regard such products as integers

$$A \cdot B \in \mathbb{Z} \cong H_0(M;\mathbb{Z}) \qquad \text{for } A \in H_k(M;\mathbb{Z}),\ B \in H_{n-k}(M;\mathbb{Z}),$$

using the canonical isomorphism $H_0(M;\mathbb{Z}) \to \mathbb{Z} : C \mapsto \langle 1, C \rangle$. As in the lectures, when $\Sigma \subset M$ is a closed oriented k-dimensional submanifold with inclusion map $i : \Sigma \hookrightarrow M$, we denote

$$[\Sigma]_M := i_*[\Sigma] \in H_k(M).$$

All homology is assumed to be with integer coefficients.[1]

For smooth maps $f : M \to M$, there is a sharp version of the Lefschetz fixed point theorem that can be stated as an intersection product calculation, namely

$$[\Gamma_f]_{M\times M} \cdot [\Delta]_{M\times M} = L(f), \tag{1}$$

where Γ_f denotes the **graph** of f,

$$\Gamma_f := \{(x, f(x)) \mid x \in M\} \subset M \times M,$$

and Δ is the **diagonal** submanifold

$$\Delta := \Gamma_{\mathrm{Id}} = \{(x,x) \mid x \in M\} \subset M \times M.$$

[1] Alternatively, it is possible to drop all orientation assumptions in this problem at the cost of using homology with $\mathbb{Z}_2$ coefficients, thus replacing $L(f)$ with the $\mathbb{Z}_2$-Lefschetz number $L_{\mathbb{Z}_2} \in \mathbb{Z}_2$, defined as an alternating sum of the traces of the maps $f_* : H_k(M;\mathbb{Z}_2) \to H_k(M;\mathbb{Z}_2)$.

We equip $\Gamma_f \subset M \times M$ with the orientation such that the embedding

$$M \overset{i}{\hookrightarrow} M \times M : x \mapsto (x, f(x))$$

defines an orientation-preserving diffeomorphism of M to Γ_f; since Δ is a special case of a graph, it inherits an orientation in the same way. In light of the obvious correspondence between $\Gamma_f \cap \Delta$ and the fixed point set of f, (1) implies the Lefschetz fixed point theorem for smooth maps; moreover, it gives a quantitative interpretation of the Lefschetz number $L(f)$ as a signed count of fixed points, under the technical condition that Γ_f and Δ intersect transversely. Our goal in this problem is to prove this formula.

(a) Prove that if $A, B, C, D \in H_*(M)$ are homology classes (with integer coefficients) whose degrees satisfy

$$|A| + |B| = n = |C| + |D|,$$

then we have the following formula for intersections of cross products of degree n in $M \times M$:

$$(A \times B) \cdot (C \times D) = \begin{cases} (-1)^{|B|}(A \cdot C)(B \cdot D) & \text{if } |B| = |C| \text{ and } |A| = |D|, \\ 0 & \text{otherwise.} \end{cases}$$

Remark: You should forgive yourself if you manage to figure out every detail of this problem except the signs, but my advice on signs is this. For most of the calculation, you'll need to keep track of degrees of cohomology classes rather than homology classes. The assumption $|A| + |B| = n = |C| + |D|$ will allow you to view those degrees also as degrees of homology classes if you prefer.

(b) For an arbitrary map $f : M \to M$ and homology classes $A, B \in H_*(M)$ of complementary degree $|A| + |B| = n$, prove the formula

$$[\Gamma_f]_{M \times M} \cdot (A \times B) = (-1)^{|A|} f_* A \cdot B \in \mathbb{Z}.$$

Hint: Our definition of the orientation on Γ_f means $[\Gamma_f]_{M \times M} = i_*[M]$. You may at some point find yourself needing to compute $i^*(\alpha \times \beta)$ for some cohomology classes $\alpha, \beta \in H^*(M)$. Try transforming the cross product into a cup product.

Recall that the Lefschetz number $L(f)$ is an alternating sum of the traces of the homomorphisms $f_* : H_k(M; \mathbb{Q}) \to H_k(M; \mathbb{Q})$ for $k = 0, \ldots, n$. Choose a basis $\{e_i\}$ of the rational vector space $H_*(M; \mathbb{Q}) = \bigoplus_{k=0}^n H_k(M; \mathbb{Q})$ consisting only of homogeneous elements. The intersection product on $H_*(M; \mathbb{Q})$ is equivalent to the intersection *form* on $H^*(M; \mathbb{Q})$ and is therefore nonsingular, so that the basis $\{e_i\}$ uniquely determines a *dual* basis $\{e_i'\}$ satisfying the condition

$$e_i \cdot e_j' = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

with the convention that numerical intersection products $A \cdot B \in \mathbb{Z}$ are understood to be 0 if $|A| + |B| \neq n$. Note that by the Künneth formula, the set of all cross products of the form $e_i \times e_j'$ forms a basis of $H_*(M \times M; \mathbb{Q})$, and so does the set of products of the form $e_i' \times e_j$.

(c) Prove the formula $[\Delta]_{M \times M} = \sum_k e_k \times e_k'$, where the sum ranges over all the chosen basis elements e_k.
Hint: By the nonsingularity of the intersection product, it suffices to check that both sides have the same intersection pairing with $e_i' \times e_j$ for every i, j.

(d) Prove (1).