Topology II — Humboldt-Universität zu Berlin
C. Wendl — Winter Semester 2024–25

PROBLEM SET 3 (extra problem)

At the end of last Friday's lecture, we sketched the construction of the long exact sequence of a simplicial pair (K, L) in simplicial homology. It derives from an algebraic result that can be expressed as follows. Suppose we have a commutative diagram of R-modules

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \xrightarrow{f} & B_n & \xrightarrow{g} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f} & B_{n-1} & \xrightarrow{g} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longrightarrow & A_{n-2} & \xrightarrow{f} & B_{n-2} & \xrightarrow{g} & C_{n-2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

in which every column is a chain complex and every row is exact. A shorter way to say this is that

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

is a short exact sequence of chain complexes and chain maps. The algebraic result states that this information determines so-called **connecting homomorphisms** $\partial_* : H_n(C_*) \to H_{n-1}(A_*)$ for each $n \in \mathbb{Z}$ such that the sequence

$$\ldots \longrightarrow H_{n+1}(C_*) \xrightarrow{\partial_*} H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\partial_*} H_{n-1}(A_*) \longrightarrow \ldots$$

is exact.

(a) Work out the details of the diagram-chasing argument to prove that the maps ∂_* exist and are independent of choices, and that the resulting sequence of homologies is exact.

(b) In the setting of (ordered or oriented) relative simplicial homology, the short exact sequence of chain complexes that arises naturally from any simplicial pair (K, L) takes the form

$$0 \longrightarrow C_*(L) \xrightarrow{i_*} C_*(K) \xrightarrow{j_*} C_*(K, L) \longrightarrow 0,$$

where $i : L \hookrightarrow K$ and $j : (K, \varnothing) \hookrightarrow (K, L)$ are the obvious inclusions, which are simplicial maps. Check that the induced chain maps i_* and j_* can also be interpreted as the natural inclusion and quotient projection respectively, and show that the resulting connecting homomorphisms $\partial_* : H_n(K, L) \to H_{n-1}(L)$ are given explicitly by the formula

$$\partial_*[c] = [\partial c],$$

where on the left hand side, $c \in C_n(K)$ is assumed to be a relative n-cycle in (K, L), meaning that ∂c belongs to $C_{n-1}(L)$. (Note that $\partial c \in C_{n-1}(L)$ is automatically a cycle since $\partial^2 = 0$, but it is not automatically a boundary in $C_*(L)$ since $c \in C_n(K)$ might not belong to $C_n(L)$.)

A **map of simplicial pairs** $f : (K, L) \to (K', L')$ is a simplicial map $f : K \to K'$ that restricts to the subcomplex L as a simplicial map $L \to L'$; for example, the identity map $K \to K$ was interpreted in part (b) above as a map of simplicial pairs $j : (K, \varnothing) \to (K, L)$, with the understanding that the relative simplicial homology $H_*(K, \varnothing)$ is by definition the same thing as the absolute simplicial homology $H_*(K)$. Any map of simplicial pairs $f : (K, L) \to (K', L')$ gives rise to a commutative diagram

$$\begin{array}{ccccc}
(L, \varnothing) & \xhookrightarrow{i} & (K, \varnothing) & \xhookrightarrow{j} & (K, L) \\
\downarrow f & & \downarrow f & & \downarrow f \\
(L', \varnothing) & \xhookrightarrow{i'} & (K', \varnothing) & \xhookrightarrow{j'} & (K', L')
\end{array}$$

in the category of simplicial pairs, which induces a similar diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(L) & \xrightarrow{i_*} & C_*(K) & \xrightarrow{j_*} & C_*(K, L) & \longrightarrow & 0 \\
& & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\
0 & \longrightarrow & C_*(L') & \xrightarrow{i'_*} & C_*(K') & \xrightarrow{j'_*} & C_*(K', L') & \longrightarrow & 0
\end{array}$$

of (ordered or oriented) simplicial homology groups and chain maps, in which both rows are short exact sequences. The following exercise then has the consequence that in this situation, there is a commutative diagram relating the long exact sequences of (K, L) and (K', L'),

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & H_n(L) & \xrightarrow{i_*} & H_n(K) & \xrightarrow{j_*} & H_n(K, L) & \xrightarrow{\partial_*} & H_{n-1}(L) & \longrightarrow & \ldots \\
& & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\
\ldots & \longrightarrow & H_n(L') & \xrightarrow{i'_*} & H_n(K') & \xrightarrow{j'_*} & H_n(K', L') & \xrightarrow{\partial_*} & H_{n-1}(L') & \longrightarrow & \ldots
\end{array}$$

We call this the **naturality** property of the long exact sequences of pairs.

(c) Given a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{f} & B_* & \xrightarrow{g} & C_* & \longrightarrow & 0 \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & A'_* & \xrightarrow{f'} & B'_* & \xrightarrow{g'} & C_* & \longrightarrow & 0
\end{array}$$

in which both rows are short exact sequences of chain complexes and the vertical arrows are chain maps, show that the induced diagram of long exact sequences

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & H_{n+1}(C_*) & \xrightarrow{\partial_*} & H_n(A_*) & \xrightarrow{f_*} & H_n(B_*) & \xrightarrow{g_*} & H_n(C_*) & \xrightarrow{\partial_*} & H_{n-1}(A_*) & \longrightarrow & \ldots \\
& & \downarrow \gamma_* & & \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* & & \downarrow \alpha_* & & \\
\ldots & \longrightarrow & H_{n+1}(C'_*) & \xrightarrow{\partial_*} & H_n(A'_*) & \xrightarrow{f'_*} & H_n(B'_*) & \xrightarrow{g'_*} & H_n(C'_*) & \xrightarrow{\partial_*} & H_{n-1}(A'_*) & \longrightarrow & \ldots
\end{array}$$

commutes.