PROBLEM SET 4

1. In this problem, we prove that $H_1(X;\mathbb{Z})$ for a path-connected space X is isomorphic to the abelianization of its fundamental group. Fix a base point $x_0 \in X$ and abbreviate $\pi_1(X) := \pi_1(X, x_0)$, so elements of $\pi_1(X)$ are represented by paths $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = x_0$. Identifying the standard 1-simplex
$$\Delta^1 := \left\{(t_0, t_1) \in \mathbb{R}^2 \;\middle|\; t_0 + t_1 = 1,\ t_0, t_1 \geqslant 0\right\}$$
with $I := [0,1]$ via the homeomorphism $\Delta^1 \to I : (t_0, t_1) \mapsto t_0$, every path $\gamma : I \to X$ corresponds to a singular 1-simplex $\Delta^1 \to X$, which we shall denote by $\tilde{h}(\gamma)$ and regard as an element of the singular 1-chain group $C_1(X;\mathbb{Z}) = \bigoplus_{\sigma\in\mathcal{K}_1(X)} \mathbb{Z}$. Show that $\tilde{h}$ has each of the following properties:

   (a) If $\gamma : I \to X$ satisfies $\gamma(0) = \gamma(1)$, then $\partial\tilde{h}(\gamma) = 0$.

   (b) For any constant path $e : I \to X$, $\tilde{h}(e) = \partial\sigma$ for some singular 2-simplex $\sigma : \Delta^2 \to X$.

   (c) For any paths $\alpha, \beta : I \to X$ with $\alpha(1) = \beta(0)$, the concatenated path $\alpha \cdot \beta : I \to X$ satisfies $\tilde{h}(\alpha) + \tilde{h}(\beta) - \tilde{h}(\alpha \cdot \beta) = \partial\sigma$ for some singular 2-simplex $\sigma : \Delta^2 \to X$.
   Hint: Imagine a triangle whose three edges are mapped to X via the paths α, β and $\alpha \cdot \beta$. Can you extend this map continuously over the rest of the triangle?

   (d) If $\alpha, \beta : I \to X$ are two paths that are homotopic with fixed end points, then $\tilde{h}(\alpha) - \tilde{h}(\beta) = \partial f$ for some singular 2-chain $f \in C_2(X;\mathbb{Z})$.
   Hint: If you draw a square representing a homotopy between α and β, you can decompose this square into two triangles.

   (e) Applying $\tilde{h}$ to paths that begin and end at the base point x_0, deduce that $\tilde{h}$ determines a group homomorphism $h : \pi_1(X) \to H_1(X;\mathbb{Z}) : [\gamma] \mapsto [\tilde{h}(\gamma)]$.

   We call $h : \pi_1(X) \to H_1(X;\mathbb{Z})$ the **Hurewicz homomorphism**. Notice that since $H_1(X;\mathbb{Z})$ is abelian, $\ker h$ automatically contains the commutator subgroup $[\pi_1(X), \pi_1(X)] \subset \pi(X)$, thus h descends to a homomorphism on the abelianization of $\pi_1(X)$,
$$\Phi : \pi_1(X)\Big/[\pi_1(X), \pi_1(X)] \to H_1(X;\mathbb{Z}).$$
   We will now show that this is an isomorphism by writing down its inverse. For each point $p \in X$, choose arbitrarily a path $\omega_p : I \to X$ from x_0 to p, and choose ω_{x_0} in particular to be the constant path. Regarding singular 1-simplices $\sigma : \Delta^1 \to X$ as paths $\sigma : I \to X$ under the usual identification of I with Δ^1, we can then associate to every singular 1-simplex $\sigma \in C_1(X;\mathbb{Z})$ a concatenated path
$$\widetilde{\Psi}(\sigma) := \omega_{\sigma(0)} \cdot \sigma \cdot \omega_{\sigma(1)}^{-1} : I \to X$$
   which begins and ends at the base point x_0, hence $\widetilde{\Psi}(\sigma)$ represents an element of $\pi_1(X)$. Let $\Psi(\sigma)$ denote the equivalence class represented by $\widetilde{\Psi}(\sigma)$ in the abelianization $\pi_1(X)/[\pi_1(X), \pi_1(X)]$. This uniquely determines a homomorphism[1]
$$\Psi : C_1(X;\mathbb{Z}) \to \pi_1(X)/[\pi_1(X), \pi_1(X)] : \sum_i m_i\sigma_i \mapsto \sum_i m_i\Psi(\sigma_i).$$

   (f) Show that $\Psi(\partial\sigma) = 0$ for every singular 2-simplex $\sigma : \Delta^2 \to X$, and deduce that Ψ descends to a homomorphism $\Psi : H_1(X;\mathbb{Z}) \to \pi_1(X)/[\pi_1(X), \pi_1(X)]$.

   (g) Show that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are both the identity map.

[1] Since $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ is abelian, we are adopting the convention of writing its group operation as addition, so the multiplication of an integer $m \in \mathbb{Z}$ by an element $\Psi(\sigma) \in \pi_1(X)/[\pi_1(X), \pi_1(X)]$ is defined accordingly.

(h) For a closed surface Σ_g of genus $g \geqslant 2$, find an example of a nontrivial element in the kernel of the Hurewicz homomorphism $\pi_1(\Sigma_g) \to H_1(\Sigma_g)$.

2. According to Hatcher, a **good** pair (X, A) is one for which the subset $A \subset X$ is closed and is a deformation retract of some neighborhood $V \subset X$ of itself. Show that the pair (X, A) with $X := [0, 1]$ and $A := \{1, 1/2, 1/3, 1/4, \ldots, 0\}$ is not good, and compare $H_1(X, A; \mathbb{Z})$ with $H_1(X/A; \mathbb{Z})$.
Hint: X/A happens to be homeomorphic to a standard pathological example that you may have seen in Topology 1—it resembles an infinite wedge sum of circles, but has a much larger fundamental group.

3. Prove that the connecting homomorphism $\partial_* : H_n(X) \to H_{n-1}(A \cap B)$ in the Mayer-Vietoris sequence in singular homology of a space $X = \mathring{A} \cup \mathring{B}$ is given by the explicit formula
$$\partial_*[a + b] = [\partial a] \qquad \text{for } a \in C_n(A),\ b \in C_n(B).$$
Use this to verify directly that the Mayer-Vietoris sequence is exact.

4. Derive from the Mayer-Vietoris sequence[2] a simple proof that there is an isomorphism $\widetilde{h}_n(X) \cong \widetilde{h}_{n+1}(\Sigma X)$ for every axiomatic homology theory h_*, every $n \in \mathbb{Z}$ and every space X, where ΣX denotes the suspension of X.

5. Use Mayer-Vietoris sequences to compute $H_*(\mathbb{T}^2; \mathbb{Z})$ by decomposing the torus $\mathbb{T}^2$ into a union of open subsets each homotopy equivalent to S^1.
Hint: There is a useful algebraic trick for turning any long exact sequence
$$\ldots \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E \longrightarrow \ldots$$
into a short exact sequence with a specific term in the middle, e.g.
$$0 \longrightarrow \operatorname{coker}(\alpha) \xrightarrow{\beta} C \xrightarrow{\gamma} \ker(\delta) \longrightarrow 0,$$
where $\operatorname{coker}(\alpha) := B/\operatorname{im}(\alpha)$. If this short exact sequence splits, one obtains from it a formula for C.

6. Use Mayer-Vietoris sequences to compute $H_*(X; \mathbb{Z})$ and $H_*(X; \mathbb{Z}_2)$, where X is

   (a) The projective plane $\mathbb{RP}^2$.

   (b) The Klein bottle.

   Hint: $\mathbb{RP}^2$ is the union of a disk with a Möbius band, and the latter admits a deformation retraction to S^1. The Klein bottle, in turn, is the union of two Möbius bands, also known as $\mathbb{RP}^2 \# \mathbb{RP}^2$.

7. Recall that given two connected topological n-manifolds X and Y, their **connected sum** $X \# Y$ is defined by deleting an open n-disk $\mathring{\mathbb{D}}^n$ from each of X and Y and then gluing $X \backslash \mathring{\mathbb{D}}^n$ and $Y \backslash \mathring{\mathbb{D}}^n$ together along an identification of their boundary spheres.

   (a) Prove that for any $k = 1, \ldots, n-2$ and any coefficient group, $H_k(X \# Y) \cong H_k(X) \oplus H_k(Y)$.
   Hint: There are two steps, as you first need to derive a relation between $H_k(X)$ and $H_k(X \backslash \mathring{\mathbb{D}}^n)$, and then see what happens when you glue $X \backslash \mathring{\mathbb{D}}^n$ and $Y \backslash \mathring{\mathbb{D}}^n$ together.

   (b) It turns out that the formula $H_{n-1}(X \# Y; \mathbb{Z}) \cong H_{n-1}(X; \mathbb{Z}) \oplus H_{n-1}(Y; \mathbb{Z})$ also holds if X and Y are both closed orientable n-manifolds with $n \geqslant 2$, and without orientability we still have $H_{n-1}(X \# Y; \mathbb{Z}_2) \cong H_{n-1}(X; \mathbb{Z}_2) \oplus H_{n-1}(Y; \mathbb{Z}_2)$. (One can deduce both results from the properties of fundamental classes in singular homology, which we will discuss later.) Find a counterexample to the formula $H_1(X \# Y; \mathbb{Z}) \cong H_1(X; \mathbb{Z}) \oplus H_1(Y; \mathbb{Z})$ where X and Y are both closed (but not necessarily orientable) 2-manifolds.

[2] Just like the long exact sequence of a pair, the Mayer-Vietoris sequence for any axiomatic homology theory h_* remains valid after replacing h_* by its reduced counterpart $\widetilde{h}_*$. This can be deduced from Proposition 28.22 in the lecture notes, and you should assume it for Problem 4.