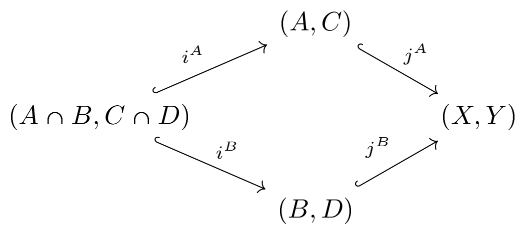
PROBLEM SET 5

1. A relative version of the Mayer-Vietoris sequence will be needed later in this course, when we prove Poincaré duality. Given pairs of spaces (X, Y), (A, C) and (B, D) such that $X = A \cup B$ and $Y = C \cup D$, the relative Mayer-Vietoris sequence in an axiomatic homology theory h_* takes the form

$$\begin{aligned} \ldots \longrightarrow h_{n+1}(X,Y) \xrightarrow{\partial_*} h_n(A\cap B, C\cap D) &\xrightarrow{(i^A_*, -i^B_*)} h_n(A,C)\oplus h_n(B,D) \\ &\xrightarrow{j^A_*\oplus j^B_*} h_n(X,Y) \xrightarrow{\partial_*} h_{n-1}(A\cap B, C\cap D) \longrightarrow \ldots, \end{aligned} \tag{1}$$

where we denote the inclusions of pairs



Let's specialize to singular homology $h_* := H_*$ (with arbitrary coefficients), and abbreviate the subcomplexes

$$C_*(A+B) := C_*(A) + C_*(B) \subset C_*(X) \qquad \text{and} \qquad C_*(C+D) := C_*(C) + C_*(D) \subset C_*(Y).$$

Notice that the inclusion $C_*(A+B) \hookrightarrow C_*(X)$ descends to a chain map $C_*(A+B, C+D) \to C_*(X,Y)$, where we define the quotient complex

$$C_*(A+B, C+D) := C_*(A+B)\Big/C_*(C+D).$$

   (a) Show that if (A, B) is an excisive couple in X and (C, D) is an excisive couple in Y, then the chain map $C_*(A+B, C+D) \to C_*(X,Y)$ induces an isomorphism on homology.
   (b) Under the same assumptions as in part (a), derive the relative Mayer-Vietoris sequence (1) in singular homology from a short exact sequence of chain complexes $0 \to C_*(A\cap B, C\cap D) \to C_*(A,C)\oplus C_*(B,D) \to C_*(A+B, C+D) \to 0$.

2. If $f : X \to X$ is a homeomorphism, then the mapping torus $X_f = (X\times[0,1])/(x,0)\sim(f(x),1)$ admits an alternative definition as

$$X_f = (X\times\mathbb{R})\Big/(x,t)\sim(f(x),t+1)$$

where the equivalence is defined for every $x \in X$ and $t \in \mathbb{R}$. Take a moment to convince yourself that these two quotients are homeomorphic. The second perspective has the advantage that one can view $\widetilde{X} := X\times\mathbb{R}$ as a covering space for X_f, with the quotient projection defining a covering map $\widetilde{X} \to X_f$ of infinite degree. Writing $S^1 := \mathbb{R}/\mathbb{Z}$, we also see a natural continuous surjective map $\pi : X_f \to S^1 : [(x,t)] \mapsto [t]$, whose **fibers** $\pi^{-1}(t)$ are homeomorphic to X for all $t \in S^1$. We shall denote by $i : X \hookrightarrow X_f$ the inclusion of the fiber $\pi^{-1}([0])$.

In lecture, we proved the existence of a long exact sequence

$$\ldots \longrightarrow h_{k+1}(X_f) \xrightarrow{\Phi} h_k(X) \xrightarrow{\mathbb{1}_* - f_*} h_k(X) \xrightarrow{i_*} h_k(X_f) \xrightarrow{\Phi} h_{k-1}(X) \longrightarrow \ldots$$

for any axiomatic homology theory h_*. The goal of this problem to gain a more concrete picture of the connecting homomorphism $\Phi : H_1(X_f;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ for the special case of singular homology with integer coefficients, under the assumption that X is path-connected and f is a homeomorphism.

Assuming X is path-connected, there is a natural isomorphism $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$, and X_f is then also path-connected. Since $H_1(X_f;\mathbb{Z})$ is isomorphic to the abelianization of $\pi_1(X_f, x)$ for any choice of base point $x \in X_f$, we can identify X with $\pi^{-1}([0]) \subset X_f$, fix a base point $x \in X \subset X_f$ and represent any class in $H_1(X_f;\mathbb{Z})$ by a loop $\gamma : [0,1] \to X_f$ with $\gamma(0) = \gamma(1) = x$. Now let $\tilde{\gamma} : [0,1] \to \widetilde{X}$ denote the unique lift of γ to the cover $\widetilde{X} = X \times \mathbb{R}$ such that $\tilde{\gamma}(0) = (x, 0)$. Since γ is a loop, it follows that $\tilde{\gamma}(1) = (f^m(x), m)$ for some $m \in \mathbb{Z}$, where f^m denotes the mth iterate of f if $m > 0$, the $(-m)$th iterate of f^{-1} if $m < 0$, and the identity map if $m = 0$.

(a) Prove that under the natural identification of $H_0(X;\mathbb{Z})$ with $\mathbb{Z}$, the connecting homomorphism $\Phi : H_1(X_f;\mathbb{Z}) \to \mathbb{Z}$ can be chosen[1] such that

$$\Phi([\gamma]) = m,$$

so in particular, $[\gamma] \in \ker \Phi$ if and only if the lift of γ to the cover $\widetilde{X}$ is a loop.

(b) Prove directly from the characterization in part (a) that $\Phi : H_1(X_f;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ is surjective. *Remark: Of course this can also be deduced less directly from the exact sequence.*

3. The Klein bottle K^2 can be presented as the mapping torus of $f : S^1 \to S^1 : e^{i\theta} \mapsto e^{-i\theta}$. Use the exact sequence of the mapping torus to compute $H_*(K^2;\mathbb{Z})$ and $H_*(K^2;\mathbb{Z}_2)$.

4. Recall that the degree $\deg(f) \in \mathbb{Z}$ of a map $f : S^0 \to S^0$ is characterized as the unique $k \in \mathbb{Z}$ such that the homomorphism

$$\mathbb{Z} \cong \widetilde{H}_0(S^0;\mathbb{Z}) \xrightarrow{f_*} \widetilde{H}_0(S^0;\mathbb{Z}) \cong \mathbb{Z}$$

is multiplication by k. In fact, there are only four possible maps $f : S^0 \to S^0$; compute the degrees of all of them.

5. Prove that for any axiomatic homology theory h_* and each $n \in \mathbb{N}$ and $x \in S^n$, the map $h_n(S^n) \to h_n(S^n, S^n \backslash \{x\})$ induced by the inclusion $(S^n, \varnothing) \hookrightarrow (S^n, S^n \backslash \{x\})$ is an isomorphism.
Hint: You can choose a neighborhood $\mathcal{U} \subset S^n$ of x homeomorphic to a disk and use $h_n(S^n, S^n \backslash \mathcal{U})$ as a substitute for $h_n(S^n, S^n \backslash \{x\})$ (why?). What kind of space is $S^n \backslash \mathcal{U}$?

6. (a) Prove that for every positive even integer n, every continuous map $f : S^n \to S^n$ has at least one point $x \in S^n$ where either $f(x) = x$ or $f(x) = -x$. Deduce that every continuous map $\mathbb{RP}^n \to \mathbb{RP}^n$ has a fixed point if n is even.

(b) Construct counterexamples to the statement in part (a) for every odd n.
Hint: Consider linear transformations with no real eigenvalues.

[1] There is a bit of freedom allowed in the definition of Φ, e.g. we could replace it with $-\Phi$ and the sequence would still be exact since $\ker \Phi$ and $\operatorname{im} \Phi$ would not change.