PROBLEM SET 6

1. Viewing S^1 as the unit circle in $\mathbb{C}$, fix a generator $[S^1] \in H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ and use it to determine local orientations $[\mathbb{C}]_z \in H_n(\mathbb{C}, \mathbb{C}\backslash\{z\};\mathbb{Z})$ for every point $z \in \mathbb{C}$ via the natural isomorphisms $H_2(\mathbb{C}, \mathbb{C}\backslash\{z\};\mathbb{Z}) \cong H_2(\mathbb{D}_z, \partial\mathbb{D}_z;\mathbb{Z}) \cong H_1(\partial\mathbb{D}_z;\mathbb{Z})$, where $\mathbb{D}_z \subset \mathbb{C}$ denotes the closed unit disk centered at z, whose boundary is canonically identified with S^1. This choice will be used in the following for the definition of local degrees of maps $f : \mathcal{U} \to \mathbb{C}$ defined on open subsets $\mathcal{U} \subset \mathbb{C}$; note that changing the generator $[S^1] \in H_1(S^1;\mathbb{Z})$ does not change the definition of $\deg(f;z)$ since it changes *both* $[\mathbb{C}]_z$ and $[\mathbb{C}]_{f(z)}$ by a sign.

   (a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is of the form $f(z) = (z - z_0)^k g(z)$ for some $z_0 \in \mathbb{C}$, $k \in \mathbb{N}$ and g a continuous map with $g(z_0) \neq 0$, then $\deg(f;z_0) = k$.

   (b) Can you modify the example in part (a) to produce one with $\deg(f;z_0) = -k$ for $k \in \mathbb{N}$?

   (c) Suppose $\mathcal{U} \subset \mathbb{C}$ is open and $f : \mathcal{U} \to \mathbb{C}$ is continuous with $f(z_0) = w_0$ and $\deg(f;z_0) \neq 0$ for some $z_0 \in \mathcal{U}$, $w_0 \in \mathbb{C}$. Prove that for any neighborhood $\mathcal{V} \subset \mathcal{U}$ of z_0, there exists an $\epsilon > 0$ such that every continuous map $\widehat{f} : \mathcal{U} \to \mathbb{C}$ satisfying $|\widehat{f} - f| < \epsilon$ maps some point in $\mathcal{V}$ to w_0.
   Remark: I have stated this problem for maps on $\mathbb{C}$ just for convenience, but one can do something similar with maps on open subsets of $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

   (d) Find an example of a continuous map $f : \mathbb{C} \to \mathbb{C}$ that has an isolated zero at the origin with $\deg(f;0) = 0$ and admits arbitrarily small continuous perturbations that are nowhere zero.
   Hint: You should probably not think in complex terms, but instead identify $\mathbb{C}$ with $\mathbb{R}^2$.

   (e) Let $f : S^2 \to S^2$ denote the natural continuous extension to $S^2 := \mathbb{C} \cup \{\infty\}$ of a complex polynomial $\mathbb{C} \to \mathbb{C}$ of degree n. What is $\deg(f)$?

   (f) Pick a constant $t_0 \in S^1$ and let $A \cong S^1 \vee S^1$ denote the subset $\{(x,y) \mid x = t_0 \text{ or } y = t_0\} \subset S^1 \times S^1 = \mathbb{T}^2$. Show that $\mathbb{T}^2/A \cong S^2$, and that the quotient map $\mathbb{T}^2 \to \mathbb{T}^2/A$ has degree ± 1 (depending on choices of generators for $H_2(\mathbb{T}^2;\mathbb{Z})$ and $H_2(S^2;\mathbb{Z})$).

2. Suppose $f : S^n \to S^n$ is any continuous map, and $p_+ \in \Sigma S^n = C_+ S^n \cup_{S^n} C_- S^n$ is the vertex of the top cone in the suspension $\Sigma S^n \cong S^{n+1}$. What is $\deg(\Sigma f; p_+)$? Use this to give a new proof (different from the one we saw in lecture) that $\deg(\Sigma f) = \deg(f)$.

3. (a) Show that every map $S^n \to \mathbb{T}^n$ has degree 0 if $n \geqslant 2$.
   Hint: Lift $S^n \to \mathbb{T}^n$ to the universal cover of $\mathbb{T}^n$.

   (b) Show that for every $d \in \mathbb{Z}$ and every $\mathbb{Z}$-admissible n-dimensional manifold M with $n \geqslant 1$, there exists a map $M \to S^n$ of degree d.
   Hint: Try a map that is interesting only on some n-ball in M and constant everywhere else.

4. For these problems you need to use the mod 2 degree, since $\mathbb{RP}^2$ and the Klein bottle are $\mathbb{Z}_2$-admissible but not $\mathbb{Z}$-admissible. (This is because they are closed and connected but not orientable).

   (a) Find an example of a map $\mathbb{RP}^2 \to S^2$ that cannot be homotopic to a constant.

   (b) Same problem but with $\mathbb{RP}^2$ replaced by the Klein bottle.
   Hint: Problem 1(f) might provide some useful inspiration.

5. The following exercise involves point-set topological issues that are related to the reason why compact subsets of CW-complexes are always contained in finite subcomplexes. The set $\mathbb{R}^\infty := \bigoplus_{j \in \mathbb{N}} \mathbb{R}$ consists of all sequences of real numbers $(x_1, x_2, x_3, \ldots)$ such that at most finitely many terms are nonzero. Identifying $\mathbb{R}^n$ for each $n \in \mathbb{N}$ with the subset

$$\left\{ (x_1, x_2, x_3, \ldots) \in \mathbb{R}^\infty \;\middle|\; x_j = 0 \text{ for all } j > n \right\},$$

we can define a topology on $\mathbb{R}^\infty$ such that a set $\mathcal{U} \subset \mathbb{R}^\infty$ is open if and only if $\mathcal{U} \cap \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ (with its standard topology) for all $n \in \mathbb{N}$.[1] Notice that every element $\mathbf{x} \in \mathbb{R}^\infty$ belongs to $\mathbb{R}^n$ for sufficiently large $n \in \mathbb{N}$. Prove that for any convergent sequence $\mathbf{x}^k \to \mathbf{x} \in \mathbb{R}^\infty$, there exists a (possibly larger) number $N \in \mathbb{N}$ such that $\mathbf{x}^k \in \mathbb{R}^N$ for all k. Deduce from this that every compact subset $K \subset \mathbb{R}^\infty$ is contained in $\mathbb{R}^N$ for some N sufficiently large (depending on K).

6. For integers $g \geqslant 0$ and $m \geqslant 1$, let $\Sigma_{g,m}$ denote the compact surface with boundary obtained by deleting m open disks from the closed oriented surface Σ_g of genus g. Assuming the isomorphism between singular and cellular homology, compute $H_*(\Sigma_{g,m}; G)$ with G an arbitrary coefficient group. *Hint: Since singular homology is homotopy invariant, you are free to replace $\Sigma_{g,m}$ by a CW-complex that is homotopy equivalent to it.*

7. The **complex projective n-space** $\mathbb{CP}^n$ is a compact $2n$-manifold defined as the set of all complex lines through the origin in $\mathbb{C}^{n+1}$, or equivalently,

$$\mathbb{CP}^n = (\mathbb{C}^{n+1}\backslash\{0\})/\sim$$

where two points $z, z' \in \mathbb{C}^{n+1}\backslash\{0\}$ are equivalent if and only if $z' = \lambda z$ for some $\lambda \in \mathbb{C}$. It is conventional to write elements of $\mathbb{CP}^n$ in so-called *homogeneous coordinates*, meaning the equivalence class represented by $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ is written as $[z_0 : \ldots : z_n]$. Notice that $\mathbb{CP}^n$ can be partitioned into two disjoint subsets

$$\mathbb{C}^n \cong \{[1 : z_1 : \ldots : z_n] \in \mathbb{CP}^n\} \quad \text{and} \quad \mathbb{CP}^{n-1} \cong \{[0 : z_1 : \ldots : z_n] \in \mathbb{CP}^n\}.$$

(a) Show that the partition $\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}$ gives rise to a cell decomposition of $\mathbb{CP}^n$ with one $2k$-cell for every $k = 0, \ldots, n$.

(b) Using the isomorphism between singular and cellular homology, compute $H_*(\mathbb{CP}^n; G)$ for an arbitrary coefficient group G. *Hint: This is easy.*

[1] This is the right topology so that the subspace topology on the set of unit vectors $S^\infty \subset \mathbb{R}^\infty$ matches the topology defined on S^∞ via the infinite-dimensional cell decomposition we discussed in lecture.