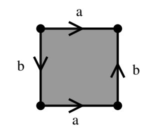
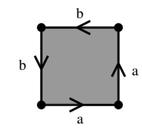
PROBLEM SET 7

1. Recall that $\mathbb{RP}^n$ has a cell decomposition with one k-cell for every $k = 0, \dots, n$, derived by starting from the cell decomposition of S^n with two cells in each dimension and dividing the whole thing by a $\mathbb{Z}_2$-action. Use this to compute $H_*^{\mathrm{CW}}(\mathbb{RP}^n; \mathbb{Z})$, $H_*^{\mathrm{CW}}(\mathbb{RP}^n; \mathbb{Z}_2)$ and $H_*^{\mathrm{CW}}(\mathbb{RP}^n; \mathbb{Q})$.

2. The picture at the right shows two spaces that you may recall from *Topologie I* are both homeomorphic to the Klein bottle. Each also defines a cell complex $X = X^0 \cup X^1 \cup X^2$ consisting of one 0-cell, two 1-cells (labeled a and b) and one 2-cell.





   (a) Compute $H_*^{\mathrm{CW}}(X; \mathbb{Z})$, $H_*^{\mathrm{CW}}(X; \mathbb{Z}_2)$ and $H_*^{\mathrm{CW}}(X; \mathbb{Q})$ for both complexes. (You'll know you've done something wrong if the answers you get from the two complexes are not isomorphic!)

   (b) Recall that the **rank** (*Rang*) of a finitely generated abelian group G is the unique integer $k \geqslant 0$ such that $G \cong \mathbb{Z}^k \oplus T$ for some finite group T. Verify for both cell decompositions of the Klein bottle above that

$$\sum_k (-1)^k \operatorname{rank} H_k^{\mathrm{CW}}(X; \mathbb{Z}) = \sum_k (-1)^k \dim_{\mathbb{Z}_2} H_k^{\mathrm{CW}}(X; \mathbb{Z}_2) = \sum_k (-1)^k \dim_{\mathbb{Q}} H_k^{\mathrm{CW}}(X; \mathbb{Q}) = 0.$$

   *(Congratulations, you've just computed the **Euler characteristic** of the Klein bottle!)*

3. Compute the cellular homology with coefficients in $\mathbb{Z}$, $\mathbb{Z}_2$ and $\mathbb{Q}$ for the following infinite-dimensional CW-complexes:

   (a) $S^\infty = \bigcup_{n=0}^\infty S^n$, whose n-skeleton for each $n \in \mathbb{N}$ is an n-sphere formed by attaching two n-cells to S^{n-1};

   (b) $\mathbb{RP}^\infty = \bigcup_{n=0}^\infty \mathbb{RP}^n = S^\infty / \mathbb{Z}_2$, whose n-skeleton for each n is the n-skeleton of S^∞ divided by a $\mathbb{Z}_2$-action;

   (c) $\mathbb{CP}^\infty = \bigcup_{n=0}^\infty \mathbb{CP}^n$, whose $2n$-skeleton for each $n \in \mathbb{N}$ is $\mathbb{CP}^n$, formed by attaching a single $2n$-cell to $\mathbb{CP}^{n-1}$.

4. Adapt the proof of $H_*^{\mathrm{CW}}(X; G) \cong h_*(X)$ we saw in lecture to prove the relative version of this statement: for any axiomatic homology theory h_* with coefficient group $h_0(\{\mathrm{pt}\}) \cong G$ and any finite-dimensional[1] CW-pair (X, A), i.e. any CW-complex X with a subcomplex $A \subset X$,

$$H_*^{\mathrm{CW}}(X, A; G) \cong h_*(X, A).$$

   Hint: Start by showing that $C_n^{\mathrm{CW}}(X, A; G)$ is canonically isomorphic to $h_n(X^n \cup A, X^{n-1} \cup A)$, and instead of the long exact sequence of the pair (X^n, X^{n-1}), consider the long exact sequence of the triple $(X^n \cup A, X^{n-1} \cup A, A)$.
   Comment: This exercise is a bit lengthy, but it is not fundamentally difficult—every step is simply a minor generalization of something that we discussed in lecture. Working through it is one of the best ways to achieve a deeper understanding of the isomorphism $H_*^{\mathrm{CW}}(X; G) \cong h_*(X)$.

[1] The assumption that (X, A) is finite dimensional is not actually necessary. The ideas needed in order to lift this assumption in the case where h_* is singular homology will be discussed next week.