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Funktionalanalysis
WiSe 2025–26

Problem Set 1

To be discussed: Thursday, 23.10.2025

Problems marked with (∗) should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1
A *Banach algebra* is a Banach space X that is equipped with the additional structure of a product $X \times X \to X : (x, y) \mapsto xy$ satisfying $\|xy\| \leqslant \|x\| \cdot \|y\|$ for all $x, y \in X$.

(a) (∗) Suppose X is a Banach space and $\mathscr{L}(X)$ denotes the Banach space of bounded linear operators $X \to X$, endowed with the operator norm. Show that $\mathscr{L}(X)$ with a product structure defined by composition $AB := A \circ B$ is a Banach algebra.

(b) (∗) Assume X is a Banach algebra containing an element $\mathbb{1} \in X$ that satisfies $\mathbb{1}x = x\mathbb{1} = x$ for all $x \in X$. Show that for any $x \in X$ with $\|x\| < 1$, the series $\sum_{n=0}^{\infty}(-1)^n x^n$ converges absolutely to an element $y \in X$ satisfying $y(\mathbb{1} + x) = (\mathbb{1} + x)y = \mathbb{1}$.

(c) Assume X and Y are Banach spaces and $A_0 \in \mathscr{L}(X, Y)$ is a continuous linear map that admits a continuous inverse $A_0^{-1} \in \mathscr{L}(Y, X)$. Find a constant $c > 0$ such that for every $A \in \mathscr{L}(X, Y)$ with $\|A - A_0\| < c$, A also has an inverse $A^{-1} \in \mathscr{L}(Y, X)$.

Problem 2
For any integer $m \geqslant 0$, let $C^m([0, 1])$ denote the vector space of m times continuously differentiable functions $x : [0, 1] \to \mathbb{R}$, with the C^m-norm

$$\|x\|_{C^m} := \sum_{k=0}^{m} \max_{t \in [0,1]} |x^{(k)}(t)|,$$

where $x^{(k)}$ denotes the kth derivative of x. Prove:

(a) $C^m([0, 1])$ is a Banach space.

(b) For each $m \geqslant 1$, the subset

$$X := \left\{ x \in C^m([0, 1]) \;\middle|\; x(0) = x'(0) = \ldots = x^{(m-1)}(0) = 0 \right\}$$

is a vector space, and endowing it with the C^m-norm makes it a Banach space.
Hint: Closed linear subspaces of Banach spaces are also Banach spaces. (Why?)

(c) (∗) The map $X \to C^0([0, 1]) : x \mapsto x^{(m)}$ is a bijective bounded linear operator with a bounded inverse.

Problem 3
Determine which (if any) of the following are closed linear subspaces of the Banach space of bounded continuous functions $f : (0, 1) \to \mathbb{R}$ with the C^0-norm:

(a) The bounded continuously differentiable functions on $(0, 1)$

(b) (∗) The uniformly continuous functions on $(0, 1)$

Problem 4
For an arbitrary topological vector space X and a seminorm $\|\cdot\|$ on X, consider the following conditions:

(i) $\|\cdot\| : X \to [0,\infty)$ is a continuous function;

(ii) The set $B_1(0) := \{x \in X \mid \|x\| < 1\} \subset X$ is open;

(iii) For every $x_0 \in X$ and $\epsilon > 0$, the set $B_\epsilon(x_0) := \{x \in X \mid \|x - x_0\| < \epsilon\} \subset X$ is open.

(a) Prove that conditions (i), (ii) and (iii) are all equivalent.
Hint: Topological vector spaces have the feature that the affine map $x \mapsto x_0 + \epsilon x$ defines a homeomorphism $X \to X$ for any $x_0 \in X$ and $\epsilon > 0$ (why?). In particular, it maps open sets to open sets.

(b) If additionally X is a locally convex space whose topology is determined by the family of seminorms $\{\|\cdot\|_\alpha\}_{\alpha\in I}$, prove that conditions (i)–(iii) are equivalent to the following: (iv) There exists a nonempty finite subset $I_0 \subset I$ and a constant $C > 0$ such that $\|x\| \leqslant C \sum_{\alpha\in I_0} \|x\|_\alpha$ for all $x \in X$.

(c) Prove that two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on a vector space V are equivalent if and only if they define the same topology.

Problem 5
Assume X is a locally convex space. Prove:

(a) A set $\mathcal{U} \subset X$ is open if and only if for every $x_0 \in \mathcal{U}$, there exists a continuous seminorm $\|\cdot\| : X \to [0,\infty)$ such that $B_1(x_0) := \{x \in X \mid \|x - x_0\| < 1\} \subset \mathcal{U}$.
Hint: Every finite positive linear combination of continuous seminorms is a continuous seminorm.

(b) X is also a topological vector space.

Problem 6 (∗)
Prove: For two locally convex spaces X and Y, a linear map $A : X \to Y$ is continuous if and only if for every continuous seminorm $\|\cdot\|_Y$ on Y, there exists a continuous seminorm $\|\cdot\|_X$ on X such that $\|Ax\|_Y \leqslant \|x\|_X$ holds for all $x \in X$.

Problem 7
Here is an example of a topological vector space whose topology cannot be defined via a metric. Let $C^0_c(\mathbb{R}^n)$ denote the space of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ that vanish outside of compact subsets.[1] We endow $C^0_c(\mathbb{R}^n)$ with a locally convex topology defined via the family of seminorms $\{\|f\|_\varphi\}_{\varphi\in I}$ where I denotes the set of all continuous functions $\varphi : \mathbb{R}^n \to [0,\infty)$ and $\|f\|_\varphi := \|\varphi f\|_{C^0}$.

(a) (∗) Show that a sequence f_j converges to f in $C^0_c(\mathbb{R}^n)$ if and only if there exists a compact set $K \subset \mathbb{R}^n$ such that $f_j|_{\mathbb{R}^n\setminus K} \equiv 0$ for every $j \in \mathbb{N} \cup \{\infty\}$ and $f_j \to f$ uniformly on K.

(b) To show that $C^0_c(\mathbb{R}^n)$ is not metrizable, one can argue by contradiction and suppose there exists a metric d such that every neighborhood $\mathcal{U} \subset C^0_c(\mathbb{R}^n)$ of 0 contains an open set of the form $B_n := \{f \in C^0_c(\mathbb{R}^n) \mid d(0,f) < 1/n\}$ for $n \in \mathbb{N}$ sufficiently large. Show that in this situation, there must exist functions $\varphi_n \in I$ such that $A_n := \{f \in C^0_c(\mathbb{R}^n) \mid \|f\|_{\varphi_n} < 1\} \subset B_n$ for every n, then derive a contradiction by constructing a neighborhood $\mathcal{U}$ of 0 that does not contain A_n for any $n \in \mathbb{N}$.

[1] We say in this case that the functions $f \in C^0_c(\mathbb{R}^n)$ have *compact support* in $\mathbb{R}^n$.