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Funktionalanalysis
WiSe 2025–26

Problem Set 10

To be discussed: Thursday, 8.01.2026

Problems marked with $(*)$ should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1
Show that every function $f \in L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant \infty$ represents a tempered distribution $\Lambda_f \in \mathscr{S}'(\mathbb{R}^n)$ defined by $\Lambda_f(\varphi) := \int_{\mathbb{R}^n} f\varphi\, dm$, and the resulting inclusion $L^p(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$ is continuous.
Hint: Use the continuity of the inclusions $\mathscr{S}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and the natural injection $L^p \hookrightarrow (L^q)^*$ for $\frac{1}{p} + \frac{1}{q} = 1$.

Problem 2
Recall that the support $\operatorname{supp}(\varphi) \subset \Omega$ of a function $\varphi : \Omega \to V$ defined on some open subset $\Omega \subset \mathbb{R}^n$ is the closure of the set of points $x \in \Omega$ at which $\varphi(x) \neq 0$. The support $\operatorname{supp}(\Lambda)$ of a distribution $\Lambda \in {}'(\Omega)$ is defined as the intersection of all closed subsets $\mathcal{U} \subset \Omega$ such that

$$\varphi \in \mathscr{D}(\Omega) \text{ with } \operatorname{supp}(\varphi) \cap \mathcal{U} = \varnothing \qquad \Rightarrow \qquad \Lambda(\varphi) = 0.$$

(a) Show that for a locally integrable function $f \in L^1_{\text{loc}}(\Omega)$ and the associated distribution $\Lambda_f \in \mathscr{D}'(\Omega)$, one has

$$\operatorname{supp}(\Lambda_f) \subset \operatorname{supp}(f),$$

and the two sets are equal if f is continuous.

(b) Find an example of a function $f \in L^1_{\text{loc}}(\Omega)$ such that $\operatorname{supp}(\Lambda_f) \neq \operatorname{supp}(f)$.
Hint: Λ_f does not change if f is modified on a set of measure zero.

(c) Show that for any distribution $\Lambda \in \mathscr{D}'(\Omega)$ and any multi-index α, $\operatorname{supp}(\partial^\alpha \Lambda) \subset \operatorname{supp}(\Lambda)$.

(d) What is the support of the Dirac delta function $\delta \in \mathscr{D}'(\mathbb{R}^n)$?

(e) $(*)$ There is a standard result in the theory of distributions stating that every distribution on $\mathbb{R}^n$ with support $\{0\} \subset \mathbb{R}^n$ is a finite linear combination of derivatives of the delta function. Use this as a black box to show that a tempered distribution has support $\{0\} \subset \mathbb{R}^n$ if and only if its Fourier transform is represented by a polynomial function on $\mathbb{R}^n$.

Problem 3 $(*)$
Recall from Problem Set 9 #1 the Cauchy-Riemann operator $\bar{\partial} := \partial_x + i\partial_y$, defined on complex-valued functions of one complex variable $z = x + iy \in \mathbb{C}$, which can equivalently be regarded as complex-valued functions of the *two real* variables $(x, y) \in \mathbb{R}^2$. In the following, we identify $\mathbb{C}$ in this way with $\mathbb{R}^2$ and endow it with the Lebesgue measure m. Consider the complex-valued function

$$f(z) := \frac{1}{2\pi z},$$

which is defined almost everywhere on $\mathbb{C}$ since $\{0\} \subset \mathbb{C}$ is a set of measure zero.

(a) Show that $f \in L^1_{\text{loc}}(\mathbb{C})$ and f represents a tempered distribution $\Lambda_f \in \mathscr{S}'(\mathbb{C})$, but f is in neither $L^1(\mathbb{C})$ nor $L^2(\mathbb{C})$.

(b) Prove that in the sense of distributions, $\bar{\partial} f = \delta$. This is equivalent to showing that for every Schwartz function $\varphi \in \mathscr{S}(\mathbb{C})$,

$$-\int_{\mathbb{C}} f \, \bar{\partial}\varphi \, dm = \varphi(0).$$

Hint: $f \, \bar{\partial}\varphi$ will be a Lebesgue-integrable function, so you can rewrite the integral as the limit as $\epsilon \to 0$ of integrals over the complement of the ϵ-ball $B_\epsilon(0) \subset \mathbb{C}$ around the origin. There are various ways to compute the integral over $\mathbb{C} \backslash B_\epsilon(0)$, e.g. you could switch to polar coordinates and use Fubini's theorem plus the fundamental theorem of calculus, or you could apply some version of Stokes' theorem.

(c) Let us write Fourier transforms of functions of $z = x + iy \in \mathbb{C}$ as functions of the complex variable $\zeta = p + iq \in \mathbb{C}$. Show that in the sense of tempered distributions, the Fourier transform $\hat{f} \in \mathscr{S}'(\mathbb{C})$ of f satisfies the equation

$$2\pi i \zeta \hat{f} = 1.$$

(d) Show that the tempered distribution $\hat{f} \in \mathscr{S}'(\mathbb{C})$ in part (c) is represented by the locally integrable function

$$\hat{f}(\zeta) = \frac{1}{2\pi i \zeta}.$$

Use as a black box the result mentioned in Problem 2(e).