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Funktionalanalysis
WiSe 2025–26

Problem Set 11

To be discussed: Thursday, 29.01.2026

Problems marked with $(*)$ should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1
Assume X is a complex Banach space and $T \in \mathscr{L}(X)$. Let us say that $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* of T if there exists a sequence $x_n \in X$ with $\|x_n\| = 1$ for all n such that $(\lambda - T)x_n \to 0$. Prove:

(a) Every approximate eigenvalue of T belongs to the spectrum $\sigma(T)$.

(b) $(*)$ If $\lambda \in \sigma(T)$ is neither an eigenvalue nor belongs to the residual spectrum of T, then it is an approximate eigenvalue of T.

(c) $(*)$ For the operator $T : \ell^1 \to \ell^1 : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$, 1 is not an eigenvalue but is an approximate eigenvalue.

Problem 2
Given a complex Banach space X and $T \in \mathscr{L}(X)$, let $T^\top \in \mathscr{L}(X^*)$ denote its transpose. Prove:

(a) If $\lambda \in \sigma(T)$ is in the residual spectrum of T then it is an eigenvalue of $T^\top$.

(b) $(*)$ If $\lambda \in \sigma(T^\top)$ is an eigenvalue of $T^\top$, then it is either an eigenvalue of T or belongs to the residual spectrum of T.
Hint: Given $(\lambda - T)x = 0$, plug $(\lambda - T)x$ into the canonical inclusion $X \hookrightarrow X^{**}$.

Now suppose X is a complex Hilbert space $\mathcal{H}$, and $T^* : \mathcal{H} \to \mathcal{H}$ denotes the adjoint operator, defined via the condition $\langle x, Ty \rangle = \langle T^* x, y \rangle$ for all $x, y \in \mathcal{H}$. Prove:

(c) $\sigma(T^*) = \left\{ \bar{\lambda} \in \mathbb{C} \;\middle|\; \lambda \in \sigma(T^\top) \right\}$

(d) $\sigma(T) = \sigma(T^\top)$

Hint: $T^* : \mathcal{H} \to \mathcal{H}$ and $T^\top : \mathcal{H}^* \to \mathcal{H}^*$ are closely related via the complex-antilinear isomorphism $\mathcal{H} \to \mathcal{H}^* : x \mapsto \langle x, \cdot \rangle$.

Problem 3
For a Lebesgue-integrable function $F : \mathbb{T}^n \to \mathbb{C}$, define the operator

$$L^2(\mathbb{T}^n) \xrightarrow{T} L^2(\mathbb{T}^n) : u \mapsto F * u, \quad \text{where} \quad (F * u)(x) = \int_{\mathbb{T}^n} F(x - y) u(y) \, dy.$$

Young's inequality (or more accurately its analogue for periodic functions) implies that T is bounded, with $\|T\| \leqslant \|F\|_{L^1}$.

(a) $(*)$ Prove that if the Fourier coefficients $\{\widehat{F}_k\}_{k \in \mathbb{Z}^n}$ of F satisfy $\lim_{|k| \to \infty} |\widehat{F}_k| = 0$, then T is compact. Show that this holds in particular if $F \in L^2(\mathbb{T}^n)$.
Hint: For inspiration, look again at the proof that the inclusions $H^s(\mathbb{T}^n) \hookrightarrow H^t(\mathbb{T}^n)$ for $s > t$ are compact.

(b) Under what assumptions on F is T a self-adjoint operator?

(c) Under what assumptions on F is T a normal operator? (cf. Problem 5)

(d) Describe the spectrum $\sigma(T)$, and find an explicit collection of eigenvectors of T that form an orthonormal basis of $L^2(\mathbb{T}^n)$. Assuming the condition in part (a), is every element of $\sigma(T)$ necessarily an eigenvalue?

Problem 4
Assume (X, μ) is a σ-finite measure space, $F : X \to \mathbb{C}$ is a bounded measurable function, and $T : L^2(X) \to L^2(X)$ is the multiplication operator $u \mapsto Fu$.

(a) Show that $\lambda \in \mathbb{C}$ belongs to the spectrum $\sigma(T)$ if and only if[1]

$$\mu\big(F^{-1}(B_\epsilon(\lambda))\big) > 0 \quad \text{for all} \quad \epsilon > 0, \tag{1}$$

where $B_\epsilon(\lambda) \subset \mathbb{C}$ denotes the open disk of radius ϵ about λ.

(b) Under what condition on F is $\lambda \in \sigma(T)$ an eigenvalue of T? When does it have finite multiplicity?

Problem 5
An operator $T \in \mathscr{L}(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$ is called *normal* if it commutes with its adjoint T^*. Prove:

(a) The following conditions on $T \in \mathscr{L}(\mathcal{H})$ are equivalent:

(i) T is normal;

(ii) $T = A + iB$ for two self-adjoint operators $A, B \in \mathscr{L}(\mathcal{H})$ that commute with each other;

(iii) $\|Tx\| = \|T^*x\|$ for every $x \in \mathcal{H}$.

Hint: Consider $\|T(x + y)\|^2$ *and* $\|T(x + iy)\|^2$ *for arbitrary* $x, y \in \mathcal{H}$.

(b) $(*)$ If T is normal, then:

(i) $\|T^2\| = \|T^*T\| = \|T\|^2$

(ii) The spectral radius of T is $\|T\|$.

(iii) Every eigenvector of T with eigenvalue $\lambda \in \mathbb{C}$ is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$. *Hint: Consider* $\|(\lambda - T)v\|^2$.

(iv) If $v, w \in \mathcal{H}$ are eigenvectors of T with distinct eigenvalues, then $\langle v, w\rangle = 0$.

(v) If T is also compact, then $\mathcal{H}$ admits an orthonormal basis consisting of eigenvectors of T.

(c) If T is *unitary* (meaning $T^*T = TT^* = \mathbb{1}$), then its spectrum is contained in the unit circle $\{|\lambda| = 1\} \subset \mathbb{C}$.
Hint: Show $\|T\| = \|T^{-1}\| = 1$, *and use the fact that operators with distance less than* 1 *from the identity map are invertible.*

[1] The set of numbers $\lambda \in \mathbb{C}$ satisfying the condition in (1) for a given function $F : X \to \mathbb{C}$ is called the *essential range* of F.