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Funktionalanalysis
WiSe 2025–26

Problem Set 12

To be discussed: Thursday, 5.02.2026

Problems marked with (∗) should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1 (∗)
For a normal operator $A \in \mathscr{L}(\mathcal{H})$ on a complex Hilbert space, prove the following without assuming the existence of a continuous functional calculus or spectral representation:

(a) A has no residual spectrum, and $\lambda \in \sigma(A)$ if and only if there exists a sequence $x_n \in \mathcal{H}$ with $\|x_n\| = 1$ for all n and $(\lambda - A)x_n \to 0$.

(b) If $P(A) \in \mathscr{L}(\mathcal{H})$ is defined by $P(A) := \sum_{j,k} a_{j,k} A^j (A^*)^k$ for polynomial functions $P : \mathbb{C} \to \mathbb{C}$ of the form $P(z) = \sum_{j,k} a_{j,k} z^j \bar{z}^k$, then $P(\sigma(A)) \subset \sigma(P(A))$.

Problem 2
For a normal operator $A \in \mathscr{L}(\mathcal{H})$ and a bounded Borel-measurable function $f : \sigma(A) \to \mathbb{C}$, prove that $\sigma(f(A))$ is contained in the closure of $f(\sigma(A))$.
Hint: If $\mu \notin \overline{f(\sigma(A))}$, then $g(z) := \frac{1}{f(z)-\mu}$ is a bounded Borel-measurable function.

Problem 3
For a fixed constant $x_0 \in \mathbb{T}^n$, let $T : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ denote the translation operator

$$(Tf)(x) := f(x + x_0).$$

This operator is unitary, and therefore cannot be compact.[1]

(a) Find an explicit spectral representation for T, i.e. a σ-finite measure space (X, μ), unitary isomorphism $U : L^2(\mathbb{T}^n) \to L^2(X)$ and bounded measurable function $F : X \to \mathbb{C}$ such that UTU^{-1} is the multiplication operator $u \mapsto Fu$.
Hint: Use Fourier series.

(b) (∗) Show that depending on the value of $x_0 \in \mathbb{T}^n$, one of the following must happen: (i) $\sigma(T)$ is a finite set consisting of eigenvalues that each have infinite multiplicity; (ii) $\sigma(T)$ is the entire unit circle in $\mathbb{C}$ and consists of a countably infinite set of eigenvalues, plus an uncountable set of points that are not eigenvalues.
Hint: The spectrum is the essential range of the function $F : X \to \mathbb{C}$ mentioned in part (a).

(c) Carry out the analogues of parts (a) and (b) for a similar translation operator on $L^2(\mathbb{R}^n)$, and show that if the shift $x_0 \in \mathbb{R}^n$ is nonzero, then the spectrum in this case is always the entire unit circle in $\mathbb{C}$ but contains no eigenvalues.

Problem 4
A bounded linear operator $A \in \mathscr{L}(\mathcal{H})$ on a complex Hilbert space is called *positive* (written "$A \geqslant 0$") if $\langle x, Ax \rangle \geqslant 0$ for all $x \in \mathcal{H}$. Prove:

[1] A Banach space isomorphism is never compact unless the space is finite dimensional. (Why not?)

(a) Positive operators are always self-adjoint.
Hint: If $\langle x, Ax\rangle$ is real then $\langle x, Ax\rangle = \langle Ax, x\rangle$. Compute $\langle x + y, A(x + y)\rangle$ and $\langle x + iy, A(x + iy)\rangle$ for arbitrary $x, y \in \mathcal{H}$.

(b) A normal operator $A \in \mathscr{L}(\mathcal{H})$ is unitary if and only if $\sigma(A) \subset S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, self-adjoint if and only if $\sigma(A) \subset \mathbb{R}$, and positive if and only if $\sigma(A) \subset [0, \infty)$.

(c) If $\langle x, Ax\rangle > 0$ for all $x \neq 0 \in \mathcal{H}$, it does not follow that $0 \notin \sigma(A)$.

If $A \geqslant 0$, then part (b) implies that the function $f(x) := \sqrt{x} \geqslant 0$ is well defined and continuous on $\sigma(A)$, so we can use the Borel functional calculus to define $\sqrt{A} := f(A) \in \mathscr{L}(\mathcal{H})$. Prove:

(d) $\sqrt{A} \geqslant 0$.

(e) $\ker \sqrt{A} = \ker A$.

Finally, we establish a special case of the polar decomposition: the following can be deduced from Theorem 18.67 in the lecture notes, but try to give a direct proof without using that theorem.

(f) Show that every invertible operator $A \in \mathscr{L}(\mathcal{H})$ factors in the form $A = UP$ where $P := \sqrt{A^*A} \geqslant 0$ and $U \in \mathscr{L}(\mathcal{H})$ is unitary.

Problem 5
Prove that a bounded linear operator $U \in \mathscr{L}(\mathcal{H})$ on a complex Hilbert space is unitary if and only if $U = e^{iA}$ for some self-adjoint operator $A \in \mathscr{L}(\mathcal{H})$.

Problem 6
The *spectral measure* μ_x corresponding to a normal operator $A \in \mathscr{L}(\mathcal{H})$ and $x \in \mathcal{H}$ is by definition the unique finite regular measure on the Borel sets in $\sigma(A) \subset \mathbb{C}$ such that

$$\langle x, f(A)x\rangle = \int_{\sigma(A)} f \, d\mu_x \quad \text{for all } f \in C^0(\sigma(A)).$$

(a) Describe μ_x explicitly in the case where $x \in \mathcal{H}$ is an eigenvector of A.

(b) Describe μ_x explicitly in the case where A is compact and $x \in \mathcal{H}$ is arbitrary.

(c) Recall that $v \in \mathcal{H}$ is called *cyclic* for A if the span of all elements of the form $A^m(A^*)^n v$ for nonnegative integers $m, n \geqslant 0$ is dense in $\mathcal{H}$. Show that if A has an eigenvalue of multiplicity greater than 1, then $\mathcal{H}$ does not contain any cyclic elements for A.
Hint: Decompose $v \in \mathcal{H}$ with respect to the splitting $\mathcal{H} = E_\lambda \oplus E_\lambda^\perp$ for $E_\lambda := \ker(\lambda - A)$, and remember that eigenvectors of A are also eigenvectors of A^*.

(d) $(*)$ Show that in the case $\mathcal{H} = \mathbb{C}^n$, the converse of part (c) also holds: if $\sigma(A)$ contains n distinct eigenvalues, then a cyclic element $v \in \mathcal{H}$ for A exists. Give an explicit example of v in the case where $A : \mathbb{C}^n \to \mathbb{C}^n$ is diagonal.
Hint: The proof of the spectral theorem will tell you where to look for an example.