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Funktionalanalysis
WiSe 2025–26

Problem Set 2

To be discussed: Thursday, 30.10.2025

Problems marked with $(*)$ should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1

Assume X is a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $V \subset X$ is a subspace. One says that V has *codimension* k if the quotient vector space X/V has dimension k. (Note that k may be finite even if X and V are both infinite dimensional.)

(a) Show that the following conditions are equivalent:

  (i) $\operatorname{codim} V = 1$;
  (ii) There exists a vector $w \in X \backslash V$ such that every $x \in X$ can be written as $x = v + \lambda w$ for unique elements $v \in V$ and $\lambda \in \mathbb{K}$;
  (iii) $V = \ker \Lambda$ for some nontrivial linear map $\Lambda : X \to \mathbb{K}$.[1]

(b) Show that if $V = \ker \Lambda = \ker \Lambda'$ for two linear functionals $\Lambda, \Lambda' : X \to \mathbb{K}$, then $\Lambda' = c\Lambda$ for some nonzero scalar $c \in \mathbb{K}$.

(c) $(*)$ Assuming X is a normed vector space and $V = \ker \Lambda$ for a nontrivial linear functional $\Lambda : X \to \mathbb{K}$, show that the following conditions are equivalent:

  (a) $V \subset X$ is closed;
  (b) $V \subset X$ is not dense;
  (c) $\Lambda : X \to \mathbb{K}$ is continuous.

  Hint: Show that the closure of any subspace is also a subspace. If $\Lambda : X \to \mathbb{K}$ is not bounded, there exists a bounded sequence $x_n = v_n + \lambda_n w \in X$ with $v_n \in V$, $w \in X \backslash V$ and $\lambda_n \in \mathbb{K}$ such that $|\Lambda(x_n)| \to \infty$. What can you say about $\frac{v_n}{\lambda_n}$?

Problem 2

Here is an example of a topological vector space that is not locally convex and has a trivial dual space. Define $L^p([0,1])$ as usual to be the space of equivalence classes (up to equality almost everywhere) of measurable functions $f : [0,1] \to \mathbb{R}$ with $\|f\|_{L^p} := \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p} < \infty$, but instead of $p \geqslant 1$, assume $0 < p < 1$. In this case, Minkowski's inequality does not hold, so $\|\cdot\|_{L^p}$ is not a norm, but we shall regard $L^p([0,1])$ as a metric space with the metric defined by $d(f,g) := \|f - g\|_{L^p}^p$.

(a) Show that d is a metric on $L^p([0,1])$.
Hint: Show first that $(x+y)^p \leqslant x^p + y^p$ holds for all $x, y \geqslant 0$. The latter can be deduced from the relation $a^q + b^q \leqslant (a+b)^q$ for $a, b \geqslant 0$ and $q := 1/p > 1$, which follows in turn from $(1+x)^q \geqslant 1 + x^q$ for $x \geqslant 0$, which you can prove by differentiating with respect to x.

[1] Linear maps $X \to \mathbb{K}$ are also called *linear functionals*, and subspaces $V \subset X$ of codimension 1 are also called *hyperplanes*.

(b) Prove that $L^p([0,1])$ with the topology defined via d is a topological vector space.

(c) Prove that the space of bounded measurable real-valued functions is dense in $L^p([0,1])$.
Hint: Given $f \in L^p([0,1])$, define functions f_n that match f wherever $|f| \leqslant n$.

(d) Prove by induction on $N \in \mathbb{N}$ that if K is any convex subset of a vector space, then for any finite collections $x_1, \ldots, x_N \in K$ and $\tau_1, \ldots, \tau_N \in [0,1]$ with $\sum_{i=1}^N \tau_i = 1$, $\sum_{i=1}^N \tau_i x_i \in K$. (This is known as a *convex combination* of $x_1, \ldots, x_N$.)

(e) Prove that for any given $\epsilon > 0$, every bounded measurable function $f : [0,1] \to \mathbb{R}$ can be written as $f = \frac{1}{N}\sum_{i=1}^N f_i$ for some finite collection of functions $f_i \in L^p([0,1])$ satisfying $d(f_i, 0) < \epsilon$ for all $i = 1, \ldots, N$. Conclude that the only closed convex subset of $L^p([0,1])$ containing a neighborhood of 0 is $L^p([0,1])$ itself.
Hint: Define each f_i to have support in an interval of length $1/N$, then make N large.

(f) Prove that all continuous linear functionals $\Lambda : L^p([0,1]) \to \mathbb{R}$ are trivial.
Hint: What kind of subset is $\{f \in L^p([0,1]) \mid |\Lambda(f)| \leqslant 1\}$?

Problem 3
Let $\mathbb{R}^\infty$ denote the vector space of infinite tuples $x = (x_1, x_2, \ldots)$ of real numbers such that at most finitely many of the coordinates $x_n \in \mathbb{R}$ are nonzero. This becomes an inner product space if we define on $\mathbb{R}^\infty$ the obvious generalization of the Euclidean inner product,

$$\langle x, y \rangle := \sum_{n=1}^{\infty} x_n y_n \in \mathbb{R}, \tag{1}$$

where the sum always converges since only finitely many of its terms are nonzero. Define a subspace $V \subset \mathbb{R}^\infty$ by $V := \left\{ x \in \mathbb{R}^\infty \;\middle|\; \sum_{n=1}^\infty \frac{x_n}{n} = 0 \right\}$.

(a) Prove that $V \subset \mathbb{R}^\infty$ is a closed subspace of codimension 1.

(b) (∗) Prove that the orthogonal complement $V^\perp = \left\{ x \in \mathbb{R}^\infty \;\middle|\; \langle x, v \rangle = 0 \text{ for all } v \in V \right\}$ is the trivial subspace of $\mathbb{R}^\infty$.

(c) In lecture we proved that for any closed subspace V in a Hilbert space $\mathcal{H}$, $\mathcal{H} = V \oplus V^\perp$. Where does the proof of this theorem go wrong if you try to carry it out with the Hilbert space $\mathcal{H}$ replaced by the *incomplete* inner product space $\mathbb{R}^\infty$?

Hint: $\mathbb{R}^\infty$ is a dense subspace of the Hilbert space ℓ^2 consisting of tuples $x = (x_1, x_2, \ldots)$ that are allowed to have infinitely many nonzero coordinates but must also satisfy $\sum_{n=1}^\infty x_n^2 < \infty$. Equivalently, ℓ^2 is $L^2(\mathbb{N}, \nu)$, the space of square-integrable functions $\mathbb{N} \to \mathbb{R} : n \mapsto x_n$ with the counting measure ν. Notice that $V = \mathbb{R}^\infty \cap z^\perp$ for an element $z \in \ell^2 \backslash \mathbb{R}^\infty$.

Problem 4
For vectors $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, consider the norms

$$|x|_p := \left(\sum_{j=1}^n x_j^p \right)^{1/p} \quad \text{for } 1 \leqslant p < \infty, \qquad |x|_\infty := \max\{|x_1|, \ldots, |x_n|\}.$$

(a) Show (by drawing pictures of the unit ball) that the normed vector spaces $(\mathbb{R}^n, |\cdot|_1)$ and $(\mathbb{R}^n, |\cdot|_\infty)$ are not strictly convex.

(b) (∗) Show that the spaces of real-valued functions of class L^1 or L^∞ on $[0,1]$ are not strictly convex.