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Funktionalanalysis
WiSe 2025–26

Problem Set 3

To be discussed: Thursday, 6.11.2025

Problems marked with ($*$) should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1 ($*$)
For $\mathcal{H}$ a Hilbert space and $X \subset \mathcal{H}$ a linear subspace with closure denoted by $\bar{X}$, prove $(X^\perp)^\perp = \bar{X}$. Does this remain true in general if $\mathcal{H}$ is assumed to be an inner product space but not complete?

Problem 2
Assume X and Y are inner product spaces, and $A : X \to Y$ and $A^* : Y \to X$ are linear maps satisfying the adjoint relation

$$\langle y, Ax \rangle = \langle A^* y, x \rangle \quad \text{for all } x \in X,\ y \in Y.$$

Denote the images of these operators by $\operatorname{im} A \subset Y$ and $\operatorname{im} A^* \subset X$.

(a) Prove: $\ker A^* = (\operatorname{im} A)^\perp$ and $\ker A = (\operatorname{im} A^*)^\perp$.

(b) ($*$) Assume Y is complete, $A : X \to Y$ is continuous and its image is closed. Show that for a given $y \in Y$, the equation $Ax = y$ has solutions $x \in X$ if and only if $\langle y, z \rangle = 0$ for all $z \in \ker A^*$.

Problem 3
For an inner product space $\mathcal{H}$ and subspace $W \subset \mathcal{H}$ such that $\mathcal{H} = W \oplus W^\perp$, the *orthogonal projection to* W is the unique linear map $P : \mathcal{H} \to \mathcal{H}$ such that $P|_W$ is the identity map on W and $\ker P = W^\perp$. Prove:

(a) P is bounded and self-adjoint,[1] and satisfies $P^2 = P$.

(b) The orthogonal projection to $W^\perp$ is given by $\mathbb{1} - P : \mathcal{H} \to \mathcal{H}$.

(c) ($*$) If $\mathcal{H}$ is complete and $\Pi : \mathcal{H} \to \mathcal{H}$ is a self-adjoint bounded linear operator with $\Pi^2 = \Pi$, then $\operatorname{im} \Pi \subset \mathcal{H}$ is closed and Π is the orthogonal projection onto $\operatorname{im} \Pi$.
Hint: The image of an orthogonal projection is the kernel of another one.

Problem 4
For a Hilbert space $\mathcal{H}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, associate to each $x \in \mathcal{H}$ the corresponding dual vector $\Lambda_x := \langle x, \cdot \rangle \in \mathcal{H}^*$.[2]

(a) Show that the formula $\langle \Lambda_x, \Lambda_y \rangle := \langle y, x \rangle$ defines an inner product on $\mathcal{H}^*$ such that the operator norm $\| \cdot \|$ satisfies $\|\Lambda\|^2 = \langle \Lambda, \Lambda \rangle$ for all $\Lambda \in \mathcal{H}^*$, thus making $\mathcal{H}^*$ into a Hilbert space over $\mathbb{K}$.

(b) Prove that every Hilbert space is reflexive.

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[1] A linear operator $L : \mathcal{H} \to \mathcal{H}$ on an inner product space is called *self-adjoint* if it satisfies $\langle x, Ly \rangle = \langle Lx, y \rangle$ for all $x, y \in \mathcal{H}$.

[2] Recall that in the case $\mathbb{K} = \mathbb{C}$, our convention is that $\langle\ ,\ \rangle$ is complex-antilinear in its first argument and complex-linear in its second. It follows that the isomorphism $\mathcal{H} \to \mathcal{H}^* : x \mapsto \Lambda_x$ is complex-antilinear.

Problem 5
Let ν denote the counting measure on a set I, i.e. every subset $E \subset I$ is ν-measurable and $\nu(E) \in \mathbb{N} \cap \{0, \infty\}$ is the number of points in E. It follows that every function $f : I \to \mathbb{C}$ is ν-measurable, and by a straightforward exercise in measure theory, a ν-integrable function can be nonzero on at most countably many points $\alpha_1, \alpha_2, \alpha_3, \ldots \in I$, so that its integral is given by an absolutely convergent series

$$\int_I f \, d\nu = \sum_{\alpha \in I} f(\alpha) := \sum_{n=1}^{\infty} f(\alpha_n) \in \mathbb{C}.$$

All summations appearing in the following should be understood in this sense. The complex Hilbert space $L^2(I, \nu)$ now consists of all functions $f : I \to \mathbb{C}$ that are nonzero on at most countably many points and satisfy $\|f\|_{L^2}^2 = \sum_{\alpha \in I} |f(\alpha)|^2 < \infty$, with the inner product of two functions in this space given by

$$\langle f, g \rangle_{L^2} = \sum_{\alpha \in I} \overline{f(\alpha)} g(\alpha) \in \mathbb{C}.$$

(a) Show that if the set I is finite or countably infinite, then $L^2(I, \nu)$ is separable.
Hint: Show that every $f \in L^2(I, \nu)$ can be approximated arbitrarily well by functions that have real and imaginary parts in $\mathbb{Q}$ at all points and are nonzero on at most finitely many.

(b) Show that if I is uncountable, then $L^2(I, \nu)$ is not separable.

(c) (*) If $\mathcal{H}$ is a complex[3] Hilbert space with orthonormal basis $\{e_\alpha\}_{\alpha \in I}$, show that the map

$$\mathcal{H} \to L^2(I, \nu) : x \mapsto f_x \quad \text{where} \quad f_x(\alpha) := \langle e_\alpha, x \rangle$$

is a unitary isomorphism of Hilbert spaces, i.e. it is an isomorphism and satisfies $\langle f_x, f_y \rangle_{L^2} = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$. Conclude that both this map and its inverse are continuous, and that $\mathcal{H}$ is separable if and only if I is not uncountable.

Comment: Almost all infinite-dimensional Hilbert spaces that one encounters in applications (e.g. $L^2(\mathbb{R})$ or $L^2([0, 1])$ and the related Sobolev spaces that we will study later) turn out to be separable. Thus all of them are unitarily isomorphic to $\ell^2 := L^2(\mathbb{N}, \nu)$.

Problem 6
For $\mathcal{H}$ a Hilbert space containing an infinite orthonormal set $e_1, e_2, e_3, \ldots \in \mathcal{H}$, prove that the bounded sequence $\{e_n\}_{n=1}^{\infty}$ has no convergent subsequence. In particular, the closed unit ball in $\mathcal{H}$ is not compact.
Comment: A topological space X is called "locally compact" if for every point $x \in X$, every neighborhood of x contains a compact neighborhood of x, e.g. in a Hilbert space, such a neighborhood could be a sufficiently small closed ball about x. Local compactness in a Hilbert space is in fact equivalent to the condition that the closed unit ball is compact, so this problem in combination with a standard result from first-year analysis proves that a Hilbert space is locally compact if and only if it is finite dimensional. We will later prove that the same is true in Banach spaces; in fact, it is true in arbitrary Hausdorff topological vector spaces. If you're curious to see a proof of the latter statement, see
`https://terrytao.wordpress.com/2011/05/24/locally-compact-topological-vector-spaces/`

[3] The analogous statement for a real Hilbert space is obtained by taking functions in $L^2(I, \nu)$ to be real valued and omitting complex conjugation from all formulas.