Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, J. Rzemieniecki

Funktionalanalysis
WiSe 2025–26

Problem Set 4

To be discussed: Thursday, 13.11.2025

Problems marked with $(*)$ should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Convention: *Unless otherwise stated, you can assume in every problem that (X, μ) is an arbitrary measure space and functions in $L^p(X) := L^p(X, \mu)$ take values in a fixed finite-dimensional inner product space $(V, \langle\ ,\ \rangle)$ over a field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. Whenever X is a subset of $\mathbb{R}^n$, you can also assume by default that μ is the Lebesgue measure m.*

Problem 1 $(*)$
Assume $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove:

(a) For any closed subspace $E \subset L^p(X)$ with $E \neq L^p(X)$, there exists a function $g \in L^q(X) \backslash \{0\}$ that satisfies $\int_X \langle g, f \rangle \, d\mu = 0$ for every $f \in E$.
Hint: Since $L^p(X)$ is uniformly convex, there exists a closest point in E to any given point in $L^p(X) \backslash E$.

(b) A linear subspace $E \subset L^p(X)$ is dense if and only if every bounded linear functional $\Lambda : L^p(X) \to \mathbb{K}$ that vanishes on E is trivial.

Comment: The result of this problem is often used in applications and cited as a consequence of the Hahn-Banach theorem, which implies a similar result for subspaces of arbitrary Banach spaces. However, the uniform convexity of $L^p(X)$ for $1 < p < \infty$ makes the use of the Hahn-Banach theorem (which relies on the axiom of choice) unnecessary in this setting. You should not use it in your solution either, since we have not proved it yet.

Problem 2
Show that if $f \in L^\infty(X)$ satisfies $|f| < \|f\|_{L^\infty}$ almost everywhere, then

$$\left| \int_X \langle g, f \rangle \, d\mu \right| < \|g\|_{L^1} \cdot \|f\|_{L^\infty} \quad \text{for every} \quad g \in L^1(X) \backslash \{0\},$$

i.e. there is *strict* inequality. Give an example $f \in L^\infty([0, 1])$ satisfying this condition.
Hint: What can you say about $\int_X (c - |f|)|g| \, d\mu$ if $|f| < c$ almost everywhere?
Comment: The Hahn-Banach theorem implies that for every nontrivial element x in a Banach space E, there exists a bounded linear functional $\Lambda \in E^*$ with $\|\Lambda\| = 1$ and $\Lambda(x) = \|x\|$. For $E = L^\infty(X)$, it follows that this $\Lambda \in E^*$ cannot be represented as $\Lambda_g = \int_X \langle g, \cdot \rangle \, d\mu$ for any $g \in L^1(X)$. This is one way of seeing that the Riesz representation theorem is false for $p = \infty$.

Problem 3

(a) Show that if (M, d) is a metric space containing an uncountable subset $S \subset M$ such that every pair of distinct points $x, y \in S$ satisfies $d(x, y) \geqslant \epsilon$ for some fixed $\epsilon > 0$, then M is not separable.

(b) Suppose (X, μ) contains infinitely many disjoint subsets with positive measure. Show that $L^\infty(X)$ contains an uncountable subset $S \subset L^\infty(X)$, consisting of functions that take only the values 0 and 1, such that $\|f - g\|_{L^\infty} = 1$ for any two distinct $f, g \in S$. Conclude that $L^\infty(X)$ is not separable.
Hint: If you've forgotten or never seen the proof via Cantor's diagonal argument that $\mathbb{R}$ is uncountable, looking it up may help.

(c) Let $\mathscr{L}(\mathcal{H})$ denote the Banach space of bounded linear operators $\mathcal{H} \to \mathcal{H}$ on a separable Hilbert space $\mathcal{H}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Show that any orthonormal basis $\{e_n\}_{n=1}^\infty$ of $\mathcal{H}$ gives rise to a continuous linear inclusion

$$\Psi : \ell^\infty \hookrightarrow \mathscr{L}(\mathcal{H}),$$

where ℓ^∞ denotes the Banach space of bounded sequences $\{\lambda_n \in \mathbb{K}\}_{n=1}^\infty$ with norm $\|\{\lambda_n\}\|_{\ell^\infty} := \sup_{n \in \mathbb{N}} |\lambda_n|$, and $\Psi(\{\lambda_n\}) \in \mathscr{L}(\mathcal{H})$ is uniquely determined by the condition $\Psi(\{\lambda_n\})e_j := \lambda_j e_j$ for all $j \in \mathbb{N}$.

Comment: It is not hard to show that every subset of a separable metric space is also separable. Since $\ell^\infty = L^\infty(\mathbb{N}, \nu)$ for the counting measure ν, parts (b) and (c) thus imply that $\mathscr{L}(\mathcal{H})$ is not separable.

Problem 4 $(*)$
This problem deals with *weak* convergence $x_n \rightharpoonup x$. Assume $\mathcal{H}$ is a separable Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with orthonormal basis $\{e_n\}_{n=1}^\infty$, and consider a sequence of the form $x_n := \lambda_n e_n \in \mathcal{H}$ for some $\lambda_n \in \mathbb{K}$. Prove:

(a) $x_n \rightharpoonup 0$ whenever the sequence λ_n is bounded.

(b) If the sequence λ_n is unbounded, then x_n is not weakly convergent.
Hint: Show that $\lim_{n\to\infty}\langle e_j, x_n\rangle = 0$ for every $j \in \mathbb{N}$ and conclude that if $x_n \rightharpoonup x$ then $x = 0$. Then associate to any subsequence with $|\lambda_{n_j}| \geqslant j$ for $j = 1, 2, 3, \ldots$ an element of the form $v = \sum_{j=1}^\infty a_j e_{n_j} \in \mathcal{H}$ such that $\langle v, x_{n_j}\rangle \nrightarrow 0$ as $j \to \infty$.
Remark: We will later use a general result called the "uniform boundedness principle" to show that weakly convergent sequences must always have bounded norms. But you should not use that result here, since we have not proved it.

(c) If $|\lambda_n| \leqslant \sqrt{n}$ for all $n \in \mathbb{N}$, then every weakly open neighborhood of $0 \in \mathcal{H}$ contains infinitely many elements of the sequence x_n.

Comment: If the weak topology on $\mathcal{H}$ were metrizable, then one could deduce from part (c) that a subsequence of $\sqrt{n}e_n$ converges weakly to 0, contradicting part (b). It follows therefore that the weak topology on an infinite-dimensional Hilbert space is not metrizable.

Problem 5
Find a sequence $f_n \in L^p(\mathbb{R})$ for $1 < p < \infty$ that converges weakly to 0 but satisfies $\|f_n\|_{L^p} = 1$ for all n, and deduce that f_n has no L^p-convergent subsequence.