Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, J. Rzemieniecki

Funktionalanalysis
WiSe 2025–26

Problem Set 5

To be discussed: Thursday, 20.11.2025

Problems marked with ($*$) should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Convention: *You can assume unless stated otherwise that all functions take values in a fixed finite-dimensional inner product space* $(V, \langle\ ,\ \rangle)$ *over a field* $\mathbb{K}$ *which is either* $\mathbb{R}$ *or* $\mathbb{C}$. *The Lebesgue measure on* $\mathbb{R}^n$ *is denoted by* m.

Problem 1
Show that the space of bounded continuous functions on $\mathbb{R}$ is not dense in $L^\infty(\mathbb{R})$.

Problem 2
Fix $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show that if $p > 1$ and $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \langle f, \varphi \rangle \, dm = 0$ for all smooth compactly supported functions $\varphi \in C_0^\infty(\mathbb{R}^n)$, then $f = 0$ almost everywhere.[1]

(b) ($*$) Assume $1 < p < \infty$, and suppose $T, T^* : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ are two linear operators satisfying the "adjoint" relation

$$\int_{\mathbb{R}^n} \langle Tf, g \rangle \, dm = \int_{\mathbb{R}^n} \langle f, T^* g \rangle \, dm \quad \text{for all} \quad f, g \in C_0^\infty(\mathbb{R}^n).$$

Show that T extends to a bounded linear operator $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ if and only if T^* extends to a bounded linear operator $T^* : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$.

Hint: Use the isometric identification of L^p with the dual space of L^q. (In part (a), this makes sense only after restricting to a compact subset.) You will also need to use the density of C_0^∞ in L^p.

Problem 3 ($*$)
Show that for any $f, g \in L^1(\mathbb{R}^n)$ and a compactly supported smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \langle \varphi * f, g \rangle \, dm = \int_{\mathbb{R}^n} \langle f, \varphi^- * g \rangle \, dm,$$

where $\varphi^-(x) := \varphi(-x)$.
Hint: Here is a useful fact about integrals of vector-valued functions. If $L : V \to W$ is a linear map between finite-dimensional vector spaces and $f : \mathbb{R}^n \to V$ is Lebesgue integrable, then $Lf : \mathbb{R}^n \to W$ is also Lebesgue integrable and $\int_{\mathbb{R}^n} Lf \, dm = L\left(\int_{\mathbb{R}^n} f \, dm\right)$.

Problem 4 ($*$)
For an integer $m \geqslant 0$, let $C_b^m(\mathbb{R}^n)$ denote the Banach space of C^m-functions $\mathbb{R}^n \to V$ whose derivatives up to order m are all bounded, with the usual C^m-norm. Let $C^m(\bar{\mathbb{R}}^n)$

[1] We will see when we study distributions that the result of Problem 2(a) is also true for $p = 1$, but that case is trickier to prove.

denote the subspace consisting of functions $f \in C_b^m(\mathbb{R}^n)$ whose derivatives of order m are also uniformly continuous.[2] One can show along the lines of Problem Set 1 #3(b) that $C^m(\bar{\mathbb{R}}^m)$ is a closed subspace of $C_b^m(\mathbb{R}^n)$, so it is also a Banach space. Prove that if $f \in C^m(\bar{\mathbb{R}}^n)$ and $\{\rho_j : \mathbb{R}^n \to [0, \infty)\}_{j \in \mathbb{N}}$ is an approximate identity with shrinking support, then

$$\lim_{j \to \infty} \|\rho_j * f - f\|_{C^m} = 0,$$

and conclude that $C^\infty(\mathbb{R}^n) \cap C^m(\bar{\mathbb{R}}^n)$ is dense in $C^m(\bar{\mathbb{R}}^n)$.
Hint: A similar (though non-identical) result is proved at the end of §8 in the lecture notes. We did not cover it in lecture.

[2]Note that for $f \in C^m(\bar{\mathbb{R}}^n)$, the derivatives of any order $k < m$ are also uniformly continuous, but this is not an extra condition; it follows (via the fundamental theorem of calculus) from the assumption that the derivatives of order $k + 1$ are bounded.