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**Funktionalanalysis**
**WiSe 2025–26**

# Problem Set 6

To be discussed: Thursday, 27.11.2025

Problems marked with (∗) should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

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**Convention**: *You can assume unless stated otherwise that all functions take values in a fixed finite-dimensional inner product space* $(V, \langle\ ,\ \rangle)$ *over a field* $\mathbb{K}$ *which is either* $\mathbb{R}$ *or* $\mathbb{C}$. *The Lebesgue measure on* $\mathbb{R}^n$ *is denoted by* $m$.

### Problem 1

Assume $f : I \to V$ is a function defined on an interval $I \subset \mathbb{R}$.

(a) (∗) Show that $f$ is Lipschitz-continuous with Lipschitz constant $C > 0$[1] if and only if there exist constants $a \in I$ and $v_0 \in V$ and a function $g \in L^\infty(I)$ with $\|g\|_{L^\infty} \leqslant C$ such that $f(x) = v_0 + \int_a^x g(t)\,dt$ for all $x \in I$.
*Hint: Start by proving directly that Lipschitz-continuity implies absolute continuity.*

(b) Find an explicit example of a function $f : [0,1] \to \mathbb{R}$ that is absolutely but not Lipschitz-continuous.

(c) (∗) One says that $f : I \to V$ has a *jump discontinuity* at $x_0 \in I$ if $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exist but are not equal. Assume $f \in L^1_{\text{loc}}(I)$ and set $F(x) := \int_a^x f(t)\,dt$ for some constant $a \in I$. Show that if $f$ has a jump discontinuity at $x_0$, then $F$ is not differentiable at $x_0$.

(d) Let $\varphi = \chi_{[0,\infty)} : \mathbb{R} \to \mathbb{R}$ denote the characteristic function of $[0,\infty)$. Given an enumeration of the rational numbers $q_1, q_2, q_3, \ldots \in \mathbb{Q}$, show that $f(x) := \sum_{n=1}^\infty \frac{1}{2^n}\varphi(x - q_n)$ defines a real-valued function $f \in L^\infty(\mathbb{R})$ such that for every $n \in \mathbb{N}$, $\lim_{x \to q_n^+} f(x) = f(q_n)$ and $\lim_{x \to q_n^-} f(x) = f(q_n) - \frac{1}{2^n}$.
*Hint: For any* $n, N \in \mathbb{N}$, *there exists a neighborhood* $J \subset I$ *of* $q_n$ *such that the function* $\varphi_m(x) := \frac{1}{2^m}\varphi(x - q_m)$ *is constant on* $J$ *for all* $m \leqslant N$ *with the exception of* $m = n$. *(Why?)*

(e) Write down an example of a Lipschitz-continuous function $F : \mathbb{R} \to \mathbb{R}$ that fails to be differentiable on a dense subset of $\mathbb{R}$. (By part (a) and the Lebesgue differentiation theorem, it will still be differentiable almost everywhere.)

### Problem 2

Determine the set of Lebesgue points for each of the following functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$:

(a) (∗) The characteristic function of $\mathbb{Q}^n \subset \mathbb{R}^n$

(b) The characteristic function of $\mathbb{R}^n \backslash \mathbb{Q}^n \subset \mathbb{R}^n$

### Problem 3

Let $\mathbb{D}^n \subset \mathbb{R}^n$ denote the unit ball, and consider a function of the form $f(x) := \frac{1}{|x|^\alpha}$ on $\mathbb{R}^n \backslash \{0\}$ for some constant $\alpha > 0$.

[1] Recall: $C > 0$ is a Lipschitz constant for $f : I \to V$ if $|f(x) - f(y)| \leqslant C|x - y|$ holds for all $x, y \in I$.

(a) For which values of $\alpha > 0$ does $f$ belong to $L^1_{\text{weak}}(\mathbb{D}^n)$, and for which of these is it also in $L^1(\mathbb{D}^n)$?

(b) For which values of $\alpha > 0$ does $f$ belong to $L^1_{\text{weak}}(\mathbb{R}^n \backslash \mathbb{D}^n)$, and for which of these is it also in $L^1(\mathbb{R}^n \backslash \mathbb{D}^n)$?

**Problem 4** ($*$)
Assume $(X, \mu)$ is a measure space and $\lambda$ is another measure that is finite and absolutely continuous with respect to $\mu$. Give a direct proof (without citing the Radon-Nikodým theorem) of the following result: for every $\epsilon > 0$, there exists $\delta > 0$ such that every measurable set $A \subset X$ satisfying $\mu(A) < \delta$ also satisfies $\lambda(A) < \epsilon$.

**Problem 5**
Our proof of the Radon-Nikodým theorem in lecture established the following slightly stronger result. Assume $\lambda$ and $\mu$ are two $\sigma$-finite measures defined on the same $\sigma$-algebra on a set $X$, and consider the space $L^1(X, \lambda + \mu)$ of measurable functions $g : X \to \mathbb{R}$ that satisfy $\int_X |g| \, d(\lambda + \mu) = \int_X |g| \, d\lambda + \int_X |g| \, d\mu < \infty$. The map $g \mapsto \int_X g \, d\lambda$ then defines a bounded linear functional $L^1(X, \lambda + \mu) \to \mathbb{R}$, so by the Riesz representation theorem, there exists a unique $h \in L^\infty(X, \lambda + \mu)$ such that

$$\int_X g \, d\lambda = \int_X hg \, d(\lambda + \mu) \quad \text{for all} \quad g \in L^1(X, \lambda + \mu).$$

**Theorem** (proved in lecture): $h : X \to \mathbb{R}$ satisfies $0 \leqslant h < 1$ outside of a subset $E \subset X$ with $\mu(E) = 0$, and the resulting function $f := \frac{h}{1-h} : X \backslash E \to [0, \infty)$ satisfies

$$\int_A f \, d\mu \leqslant \lambda(A) \quad \text{for all measurable sets } A \subset X, \tag{1}$$

with equality for all mesurable sets $A \subset X$ whenever $\lambda \ll \mu$.
In the following, we consider pairs of measures $\lambda$ and $\mu$ on certain $\sigma$-algebras on sets $X \subset \mathbb{R}^n$, where each of $\lambda$ and $\mu$ is either the Lebesgue measure $m$, the counting measure $\nu$, or the Dirac measure $\delta$.[2] In each case, find the function $f : X \backslash E \to [0, \infty)$ explicitly, determine the collection of measurable subsets $A \subset X$ on which (1) becomes an equality, and determine whether $\lambda \ll \mu$.

(a) ($*$) $X = \mathbb{R}^n$, $\mu = m$ and $\lambda = \delta$ on the $\sigma$-algebra of Lebesgue-measurable sets

(b) $X = \mathbb{R}^n$, $\mu = \delta$ and $\lambda = m$ on the $\sigma$-algebra of Lebesgue-measurable sets

(c) ($*$) $X = \mathbb{Z}^n$, $\mu = \nu$ and $\lambda = \delta$ on the $\sigma$-algebra of all subsets

(d) $X = \mathbb{Z}^n$, $\mu = \delta$ and $\lambda = \nu$ on the $\sigma$-algebra of all subsets

One last brainteaser:

(e) For $X = \mathbb{R}^n$ with $\mu = \nu$ and $\lambda = m$ defined on the $\sigma$-algebra of Lebesgue-measurable sets, show that $\lambda \ll \mu$ but the function $f$ is identically zero, so (1) cannot typically be an equality. What went wrong?

[2] Recall: $\nu(A)$ is the number of elements in $A$, while $\delta(A)$ is defined to be 1 if $0 \in A$ and 0 otherwise.