Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, J. Rzemieniecki

Funktionalanalysis
WiSe 2025–26

Problem Set 7

To be discussed: Thursday, 4.12.2025

Problems marked with (∗) should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Convention: *You can assume unless stated otherwise that all functions take values in a fixed finite-dimensional inner product space* $(V, \langle\ ,\ \rangle)$ *over* $\mathbb{C}$.

Problem 1

Assume $f : \mathbb{T}^n \to \mathbb{C}$ is of class L^2.

(a) What condition on the Fourier coefficients $\{\hat{f}_k \in \mathbb{C}\}_{k \in \mathbb{Z}^n}$ is equivalent to the condition that f is a *real*-valued function?

(b) Show that if f is real-valued, then it can be presented as an L^2-convergent series[1]

$$f(x) = \sum_{k \in \mathbb{Z}^n} \left[a_k \cos(2\pi k \cdot x) + b_k \sin(2\pi k \cdot x) \right]$$

with uniquely determined real coefficients $a_k, b_k \in \mathbb{R}$ that satisfy $a_{-k} = a_k$ and $b_{-k} = -b_k$ for all $k \in \mathbb{Z}^n$ and are square summable, i.e. the functions $k \mapsto a_k$ and $k \mapsto b_k$ belong to $\ell^2(\mathbb{Z}^n)$. Write down explicit formulas for a_k and b_k as integrals.

(c) Under what conditions on a real-valued function f does the trigonometric series in part (b) contain only cosine terms or only sine terms?

(d) Show that for $n = 1$, the real-valued functions $\varphi_0(x) := 1$, $\varphi_k(x) := \sqrt{2}\cos(2\pi k x)$ and $\psi_k(x) := \sqrt{2}\sin(2\pi k x)$ for $k \in \mathbb{N}$ form an orthonormal basis of the space of real-valued L^2-functions on $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.

Problem 2

This is just Problem 1, but for Fourier transforms instead of Fourier series. Assume for simplicity that $f : \mathbb{R}^n \to \mathbb{C}$ is in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, since this will force all integrals to converge.

(a) What condition on the Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ is equivalent to the condition that f is real-valued?

(b) Show that if f is real-valued, then

$$f(x) = \int_{\mathbb{R}^n} \left[u(p) \cos(2\pi p \cdot x) + v(p) \sin(2\pi p \cdot x) \right] dp$$

for uniquely determined real-valued functions $u, v \in \mathscr{S}(\mathbb{R}^n)$ such that u is even and v is odd. Write down formulas for u and v as integrals. Under what conditions on f does one obtain $u \equiv 0$ or $v \equiv 0$?

---

[1] In the context of real-valued functions, this trigonometric series is also called the "Fourier series" of f.

Problem 3
Each of the following real-valued functions on the interval $[-1/2, 1/2)$ has a unique extension to a (not necessarily continuous) function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Compute explicitly the Fourier expansions $\sum_{k\in\mathbb{Z}} e^{2\pi ikx} \widehat{f}_k$ of each function f, and rewrite them in the form $\sum_{k=0}^{\infty} a_k \cos(2\pi kx) + \sum_{k=1}^{\infty} b_k \sin(2\pi kx)$ with real coefficients $a_k, b_k \in \mathbb{R}$. In each case, either prove that the series converges to $f(x)$ for every $x \in \mathbb{R}$ or find a specific point $x \in \mathbb{R}$ where it does not converge to $f(x)$.[2]

(a) $(*)$ The sawtooth wave: $f(x) = x$

(b) The square wave: $f(x) = \begin{cases} 1 & \text{for } 0 \leqslant x < 1/2 \\ -1 & \text{for } -1/2 \leqslant x < 0 \end{cases}$

(c) $(*)$ The triangle wave: $f(x) = |x|$

Problem 4 $(*)$
Prove that the space $\mathscr{S}(\mathbb{Z}^n)$ of rapidly decreasing functions on the lattice $\mathbb{Z}^n$ is dense in $\ell^p(\mathbb{Z}^n)$ for every $p \in [1, \infty)$, but not for $p = \infty$.

Problem 5 $(*)$
Prove the claim (stated in lecture) that the following two conditions on a pair of functions $f, g \in L^2(\mathbb{R}^n)$ are equivalent:

(i) g is equal almost everywhere to the Fourier transform of f;

(ii) There exists a sequence $R_j \to \infty$ such that $g(p) = \lim_{j\to\infty} \int_{B_{R_j}(0)} e^{-2\pi ip\cdot x} f(x)\, dx$ for almost every $p \in \mathbb{R}^n$.
Hint: We are not assuming $f \in L^1(\mathbb{R}^n)$, so the integral $\int_{\mathbb{R}^n} e^{-2\pi ip\cdot x} f(x)\, dx$ might not be well defined. However, the product of f with the characteristic function of $B_R(0)$ is in $L^1(\mathbb{R}^n)$ for every $R > 0$.

Problem 6
In this problem, we consider pairs of functions f and g for which pointwise products $f(x)g(x)$ are well defined, e.g. f can be vector valued and g scalar valued, or vice versa. Use Fubini's theorem to prove the following relations between Fourier transforms/series and convolutions:

(a) $(*)$ For $f, g \in L^1(\mathbb{R}^n)$, the Fourier transform of $f * g$ is given by $\widehat{f * g}(p) = \widehat{f}(p)\widehat{g}(p)$ for all $p \in \mathbb{R}^n$.

(b) For $f, g \in L^1(\mathbb{T}^n)$, the Fourier series of $f * g$ has coefficients $\widehat{f * g}_k = \widehat{f}_k \widehat{g}_k$ for $k \in \mathbb{Z}^n$.[3]

(c) For two continuous fully periodic functions f, g whose Fourier coefficients satisfy $\widehat{f}, \widehat{g} \in \ell^1(\mathbb{Z}^n)$, the Fourier series of fg has coefficients $\widehat{fg}_k = \sum_{j\in\mathbb{Z}^n} \widehat{f}_{k-j}\widehat{g}_j$ for $k \in \mathbb{Z}^n$.[4]

(d) Use a density argument to extend the relation in part (c) to the case where f satisfies the same hypothesis but g is an arbitrary function in $L^2(\mathbb{T}^n)$.

[2]All three functions are in $L^2(\mathbb{T}^1)$, so their Fourier series will converge in L^2 no matter what, but possibly not pointwise.

[3]The convolution of two functions on $\mathbb{T}^n$ is defined via the obvious formula $(f * g)(x) := \int_{\mathbb{T}^n} f(x - y)g(y)\, dy$. The proof of Young's inequality can be adapted almost verbatim to the fully periodic setting in order to show that $f * g \in L^1(\mathbb{T}^n)$ whenever $f, g \in L^1(\mathbb{T}^n)$.

[4]The right hand side of this relation could also be written as $(\widehat{f} * \widehat{g})_k$ after defining the convolution of two functions on $\mathbb{Z}^n$ in the obvious way as an integral with respect to the counting measure. The proof of Young's inequality also adapts to this setting, so that $f * g \in \ell^1(\mathbb{Z}^n)$ for $f, g \in \ell^1(\mathbb{Z}^n)$.