Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, J. Rzemieniecki

Funktionalanalysis
WiSe 2025–26

Problem Set 8

To be discussed: Thursday, 11.12.2025

Problems marked with $(*)$ should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1
Fix $s \geqslant 0$ and a multi-index α of order $m := |\alpha| \in \mathbb{N}$.

(a) Use the Fourier transform and Fourier inverse operators $\mathscr{F}, \mathscr{F}^* : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ to write down an explicit formula for the unique extension of $\partial^\alpha : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ to a bounded linear operator $\partial^\alpha : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$.

(b) $(*)$ Show that a sequence $f_j \in H^m(\mathbb{R}^n)$ converges in the H^m-norm to $f \in H^m(\mathbb{R}^n)$ if and only if $\partial^\beta f_j \to \partial^\beta f$ in the L^2-norm for all multi-indices β of order $|\beta| \leqslant m$.

(c) $(*)$ Show that for any scalar-valued function $f \in L^1(\mathbb{R}^n)$, the convolution with f defines a bounded linear operator $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n) : g \mapsto f * g$, and if $g \in H^{s+m}(\mathbb{R}^n)$, then $\partial^\alpha(f * g) = f * \partial^\alpha g$.
Hint: Problem Set 7 #6 proves the formula $\widehat{f * g} = \hat{f}\hat{g}$ for $f, g \in L^1(\mathbb{R}^n)$. For the present problem, you may assume this formula also holds when $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$; this case was omitted from the lecture due to lack of time, but is proved via an easy density argument as Theorem 11.18 in the lecture notes.

(d) Show that for any $f \in H^m(\mathbb{R}^n)$ and any approximate identity $\rho_j : \mathbb{R}^n \to [0, \infty)$ with shrinking support, the functions $f_j := \rho_j * f$ are in $H^m(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and converge in the H^m-norm to f as $j \to \infty$.

Problem 2
Prove that for every $m \in \mathbb{N}$, functions $f \in C^m(\mathbb{T}^n)$ are also in $H^m(\mathbb{T}^n)$ and the inclusion $C^m(\mathbb{T}^n) \hookrightarrow H^m(\mathbb{T}^n)$ is continuous.

Problem 3
Suppose $\rho \in \mathscr{S}(\mathbb{R}^n)$ satisfies $\rho \geqslant 0$ and $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$, and define $\rho_j(x) := j^n \rho(jx)$ for $j \in \mathbb{N}$.

(a) $(*)$ Show that for any $s \geqslant 0$ and $f \in H^s(\mathbb{R}^n)$, the sequence $\rho_j * f \in C^\infty(\mathbb{R}^n)$ satisfies

$$\|\rho_j * f\|_{H^s} \leqslant \|f\|_{H^s} \qquad \text{and} \qquad \rho_j * f \xrightarrow{H^s} f \text{ as } j \to \infty.$$

Note that the convergence does not follow from Problem 1(d) since we are not assuming $s \in \mathbb{N}$.

(b) Show that the same result holds if $\rho_j \in \mathscr{S}(\mathbb{R}^n)$ is instead defined as $\mathscr{F}^*\psi_j$ for a sequence of smooth functions $\psi_j : \mathbb{R}^n \to [0, 1]$ with compact support in the increasingly large ball $B_{j+1}(0) \subset \mathbb{R}^n$ and $\psi_j|_{B_j(0)} \equiv 1$.

Problem 4
Prove that for $s > t$, the natural inclusion $H^s(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$ is not a compact operator.

Problem 5
Fix constants $a, b > 1$ with $b \in \mathbb{N}$ and consider the periodic function $f(x) := \sum_{k=0}^\infty \frac{1}{a^k} e^{2\pi i b^k x}$, which is continuous since the series converges absolutely and uniformly. Prove:

(a) ($*$) $f \in H^s(S^1)$ if and only if $s < \log_b a$.

(b) $f \in C^{0,\alpha}(S^1)$ for every $\alpha \in (0,1)$ with $\alpha \leqslant \log_b a$.
Hint: Use Lemma 12.36 from the lecture notes. Note that the partial sums are continuously differentiable: estimate their $C^{0,1}$-norms.

Remark: f is a variant of the function famously introduced by Weierstrass in 1872, which is of class C^1 if $b < a$, but nowhere differentiable if $b \geqslant a$. Part (a) establishes a weak version of the latter statement by proving $f \notin H^1(S^1)$, which implies $f \notin C^1(S^1)$ via Problem 2. Notice that while the Sobolev embedding theorem provides a continuous inclusion $H^s(S^1) \hookrightarrow C^{0,\alpha}(S^1)$ whenever $\alpha \leqslant s - 1/2$, f turns out to be in a wider range of Hölder spaces than is guaranteed by that theorem. (That is just a coincidence—there is no interesting phenomenon behind it that I am aware of.)

Problem 6
Assume Ω is a compact subset of either $\mathbb{R}^n$ or $\mathbb{T}^n$, and $0 < \alpha < \beta \leqslant 1$.

(a) Prove via the Arzelà-Ascoli theorem that the inclusion $C^{0,\beta}(\Omega) \hookrightarrow C^0(\Omega)$ is compact.

(b) Show that if $f_k \in C^{0,\beta}(\Omega)$ is a uniformly $C^{0,\beta}$-bounded sequence that is C^0-convergent to $f \in C^0(\Omega)$, then f is also in $C^{0,\beta}(\Omega)$.
Caution: Do not try to prove that f_k is also $C^{0,\beta}$-convergent to f—that is not generally true.

(c) Show that the inclusion $C^{0,\beta}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ is also compact.
Hint: Given $f_k \to f$ as in part (b), use the relation

$$\frac{|g(x) - g(y)|}{|x - y|^\alpha} = \left(\frac{|g(x) - g(y)|}{|x - y|^\beta}\right)^{\alpha/\beta} \cdot |g(x) - g(y)|^{1 - \frac{\alpha}{\beta}}.$$

for the functions $g := f - f_k$.