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Funktionalanalysis
WiSe 2025–26

Problem Set 9

To be discussed: Thursday, 18.12.2025

Problems marked with (∗) should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1
The linear inhomogeneous Cauchy-Riemann equation for functions $f : \mathbb{C} \to \mathbb{C}$ of a complex variable $z = x + iy$ is a first-order PDE taking the form

$$\bar{\partial} f := \partial_x f + i\partial_y f = g.$$

Use the coordinates (x, y) to identify $\mathbb{C}$ with $\mathbb{R}^2$ and consider functions that are fully periodic on $\mathbb{R}^2$; these are equivalent to complex-valued functions on the torus $\mathbb{T}^2$. Prove:

(a) (∗) If $f \in H^1(\mathbb{T}^2)$ and $g = \bar{\partial} f \in H^m(\mathbb{T}^2)$ for some $m \in \mathbb{N}$, then $f \in H^{m+1}(\mathbb{T}^2)$.

(b) If $f \in C^1(\mathbb{T}^2)$ and $g = \bar{\partial} f$ is smooth, then f is smooth.

Problem 2 (∗)
Consider the locally integrable real-valued function $f(x) := |x|$ on $\mathbb{R}$.

(a) Prove that f has weak[1] derivative $f'(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0 \end{cases}$.

(b) Prove that f' is not weakly differentiable, but its derivative in the sense of distributions is $2\delta \in \mathscr{D}'(\mathbb{R})$.

Problem 3 (∗)
Consider the real-valued function $f(x) := \ln|x|$ on $\mathbb{R}$. The classical derivative of f away from the point $x = 0$ is the function $1/x$, which unfortunately is not integrable on domains containing the origin. Show however that f is in $L^1_{\text{loc}}(\mathbb{R})$ and its distributional derivative $\Lambda'_f \in \mathscr{D}'(\mathbb{R})$ is[2]

$$\Lambda'_f(\varphi) = \text{p. v.} \int_{\mathbb{R}} \frac{\varphi(x)}{x}\, dx := \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x}\, dx \quad \text{for} \quad \varphi \in \mathscr{D}(\mathbb{R}).$$

Comment: This is the closest one can get to saying that the weak derivative of $\ln|x|$ *is* $1/x$*, despite the latter not being in* $L^1_{\text{loc}}(\mathbb{R})$*.*

Problem 4
For $m \in \mathbb{N}$, $1 \leq p \leq \infty$ and an open domain $\Omega \subset \mathbb{R}^n$, the Sobolev space $W^{m,p}(\Omega)$ is defined as the space of equivalence classes (defined almost everywhere) of functions $f \in L^p(\Omega)$ having weak derivatives $\partial^\alpha f$ that are also in $L^p(\Omega)$ for all multi-indices α of order $|\alpha| \leq m$. The norm

$$\|f\|_{W^{m,p}} := \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p}$$

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[1] Note that there is no need to define $f'(0)$ in Problem 1(a) since $\{0\} \subset \mathbb{R}$ is a set of measure zero.

[2] The notation p. v. in Problem 2 stands for "Cauchy principal value" and is defined as the limit on the right hand side. The limit is necessary since $1/x$ is not a locally integrable function and thus $x \mapsto \varphi(x)/x$ is not always in $L^1(\mathbb{R})$ for $\varphi \in \mathscr{D}(\mathbb{R})$.

makes $W^{m,p}(\Omega)$ into a Banach space, and we denote by $W^{m,p}_{\text{loc}}(\Omega)$ the space of functions (defined almost everywhere) on $\Omega \subset \mathbb{R}^n$ whose restrictions to every open subset $\mathcal{U} \subset \Omega$ with compact closure are in $W^{m,p}(\mathcal{U})$. Prove:

(a) (∗) If f is an absolutely continuous function on an interval $[a,b]$, then its classical derivative f' (defined almost everywhere) is also its weak derivative on the domain (a,b), hence $f \in W^{1,1}((a,b))$.
Hint: For any $\varphi \in \mathscr{D}((a,b))$, φf defines an absolutely continuous function on $[a,b]$ that vanishes at the end points.

(b) If $f \in W^{1,1}_{\text{loc}}(\Omega)$ for an open subset $\Omega \subset \mathbb{R}$, then on every compact subinterval $[a,b] \subset \Omega$, f is equal almost everywhere to an absolutely continuous function.
Hint: Compare the weak derivatives of f and the function $g(x) := \int_a^x f'(t)\,dt$ on $[a,b]$.

(c) (∗) Part (b) implies that every $f \in W^{1,1}(\Omega)$ on an open interval $\Omega \subset \mathbb{R}$ can be assumed continuous after changing its values on a set of measure zero. Assuming this modification has been made, prove that there exists a constant $c > 0$ independent of f such that

$$\|f\|_{C^0} \leqslant c\|f\|_{W^{1,1}} \quad \text{for all} \quad f \in W^{1,1}(\Omega).$$

In other words, there is a continuous inclusion $W^{1,1}(\Omega) \hookrightarrow C^0_b(\Omega)$.
Hint: Prove that $|f(x) - f(y)| \leqslant \|f'\|_{L^1}$ for all $x, y \in \Omega$, and deduce from this that $|f(x)| \geqslant \|f\|_{C^0} - \|f'\|_{L^1}$ for all $x \in \Omega$.

(d) Show that for $\Omega = (-1,1)$, the continuous inclusion $W^{1,1}(\Omega) \hookrightarrow C^0(\Omega)$ in part (c) is not compact.
Hint: Describe (by drawing a picture) an L^1-convergent sequence of smooth functions $f_j : (-1,1) \to \mathbb{R}$ such that $\|f_j'\|_{L^1}$ is bounded but the L^1-limit is discontinuous.

Comment: The Sobolev embedding theorem gives continuous inclusions $W^{k,p} \hookrightarrow C^0$ when $kp > n$ with domains $\Omega \subset \mathbb{R}^n$, but no such inclusion exists in general for the so-called "Sobolev borderline cases" where $kp = n$, of which $W^{1,1}$ on $\Omega \subset \mathbb{R}$ is an example. For this reason, the result of part (c) is slightly surprising, though part (d) implies that there is no improved inclusion $W^{1,1} \hookrightarrow C^{0,\alpha}$ for any $\alpha > 0$. If there were, then $W^{1,1} \hookrightarrow C^0$ would be compact on bounded intervals $\Omega \subset \mathbb{R}$ due to the compactness of $C^{0,\alpha} \hookrightarrow C^0$, which follows from Arzelà-Ascoli.

Problem 5

When Ω is a nonempty bounded interval $(a,b) \subset \mathbb{R}$, the Sobolev embedding theorem gives continuous inclusions

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) \quad \text{if} \quad 0 < \alpha < 1,\ 1 < p \leqslant \infty \text{ and } \alpha \leqslant 1 - \frac{1}{p}$$
$$W^{2,1}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) \quad \text{if} \quad 0 < \alpha < 1.$$

Without citing the theorem, prove this as follows:

(a) Deduce the inclusions $W^{2,1} \hookrightarrow C^{0,\alpha}$ for $\alpha \in (0,1]$ from a continuous inclusion $W^{2,1} \hookrightarrow C^1$ using Problem 4.

(b) Deduce the inclusion $W^{1,p} \hookrightarrow C^0$ for every $p \geqslant 1$ from Problem 4.

(c) (∗) For $a \leqslant x < y \leqslant b$, the fundamental theorem of calculus implies $|f(x) - f(y)| \leqslant \|f'\|_{L^1([x,y])}$ for $f \in W^{1,p}(\Omega)$ since (by Problem 4) f can be assumed absolutely continuous. Use Hölder's inequality to deduce a Hölder-type estimate $|f(x) - f(y)| \leqslant c\|f'\|_{L^p} \cdot |x-y|^\alpha$ for $0 < \alpha \leqslant 1 - 1/p$ whenever $p > 1$.