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Funktionalanalysis
WiSe 2025–26

Take-Home Midterm

Due: Thursday, 22.01.2026 by 13:15 (100pts total)

Instructions

The purpose of this assignment is three-fold:

- In the absence of regular problem sets for the next two weeks, it deals with current material from the lectures (especially Problems 1 and 4).
- It gives the instructors a chance to gauge your understanding more directly than usual and give feedback.
- It provides an opportunity to improve your final grade in the course.

Concerning the first point: if you are in the habit of working through the problem sets regularly, then we strongly recommend that you work through and hand in this one as well, even if you know you cannot solve enough problems to have an impact on your grade. This pertains especially to Problems 1 and 4 since they involve material that has not been covered on any problem sets so far; the results in Problem 4, in particular, should be considered essential material that you will be expected to understand on the final exam.

To receive feedback and/or credit, you must upload your solutions to the moodle by **Thursday, January 22 at 13:15**. The solutions will be discussed in the Übung on that day.

You are free to use any resources at your disposal and to discuss the problems with your comrades, but **you must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

There are 100 points in total; a score of 60 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus. Note that the number of points assigned to each part of each problem is meant to be approximately proportional to its conceptual importance/difficulty.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. if you need to choose a smooth cutoff function with particular properties and can justify its existence with a convincing picture instead of an explicit formula (use your own judgement). Unless stated otherwise, you are free to make use of all results proved in the lecture notes—including results that were only briefly sketched in the actual lectures—as well as results proved in previous problem sets, without reproving them. When using a result from a problem set or the lecture notes, say explicitly which one.

One more piece of general advice: if you get stuck on one part of a problem, it may often still be possible to move on and do the next part.

You are free to ask for clarification or hints via e-mail/moodle or in office hours; of course we reserve the right not to answer such questions.

Problem 1 [15pts total]
The following statement is a corollary of the Hahn-Banach theorem, but has the advantage that it can be proved constructively, without invoking the axiom of choice.

Theorem: *Suppose X is a separable real Banach space, $V \subset X$ is a linear subspace, and $\lambda : V \to \mathbb{R}$ is a bounded linear functional. Then there exists a bounded linear functional $\Lambda : X \to \mathbb{R}$ such that $\Lambda|_V = \lambda$ and $\|\Lambda\| = \|\lambda\|$.*

(a) Use separability to show that there exists a nested sequence of subspaces $V = V_0 \subset V_1 \subset V_2 \subset \ldots \subset X$ such that $\dim(V_k/V_{k-1}) = 1$ for every $k \in \mathbb{N}$ and $\bigcup_{k=0}^{\infty} V_k$ is a dense subspace of X. [5pts]
Note: A similar trick was used when we proved in lecture that separable Hilbert spaces admit countable orthonormal bases.

(b) Given a sequence of subspaces $V = V_0 \subset V_1 \subset V_2 \subset \ldots \subset X$ as in part (a) and choices of elements $y_k \in V_k \backslash V_{k-1}$ for each $k \in \mathbb{N}$, describe a deterministic algorithm[1] for constructing an extension Λ of λ as claimed by the theorem. You should avoid using Zorn's lemma or the axiom of choice, but may feel free to reuse any other details from the proof of the Hahn-Banach theorem that seem helpful. [10pts]

Problem 2 [20pts total]
For integers $N \geqslant 2$, consider the sequence of functions $f_N : \mathbb{R} \to \mathbb{C}$ defined by

$$f_N(x) := \int_2^N \frac{e^{2\pi i p x}}{p \ln p} \, dp.$$

(a) Show that for each $N \geqslant 2$, the function f_N is smooth, has bounded derivatives of all orders, and belongs to $L^2(\mathbb{R})$. [7pts]

(b) Show that as $N \to \infty$, the sequence f_N is L^2-convergent to a function $f \in L^2(\mathbb{R})$, but there exists a point $x \in \mathbb{R}$ such that the sequence $f_N(x)$ is unbounded and thus not convergent. [6pts]

(c) For which real numbers $s \geqslant 0$ does the function $f \in L^2(\mathbb{R})$ in part (b) belong to the Sobolev space $H^s(\mathbb{R})$? [7pts]
Remark: No need to prove this, but part (b) is meant to hint at the fact that $f \in L^2(\mathbb{R})$ is not a bounded continuous function on $\mathbb{R}$. You should check that your answer to part (c) is consistent with this information.

Problem 3 [22pts total]
Consider a linear differential operator of the form $L = \sum_\alpha c_\alpha \partial^\alpha$ acting on real-valued functions on $\mathbb{R}^n$, where the coefficients c_α are real constants and the sum runs over finitely many multi-indices, which may be of various orders. Given $f \in \mathscr{D}'(\mathbb{R}^n)$, the partial differential equation $Lu = f$ is said to be satisfied *in the sense of distributions* if $u \in \mathscr{D}'(\mathbb{R}^n)$ is a distribution such that the distribution $\sum_\alpha c_\alpha \partial^\alpha u$ is the same as f. If u and f are locally integrable functions whose corresponding distributions satisfy this condition, we call u a *weak solution* to the PDE.

(a) Show that for any $f \in L^1_{\text{loc}}(\mathbb{R})$, the function $u(t, x) := f(t \pm x)$ (with either choice of sign) is in $L^1_{\text{loc}}(\mathbb{R}^2)$ and is a weak solution to the wave equation $\partial_t^2 u - \partial_x^2 u = 0$. [10pts]

A function $K \in L^1_{\text{loc}}(\mathbb{R}^n)$ is called a *fundamental solution*[2] for the operator L if it satisfies

[1] The word "deterministic" means that the procedure you describe should not involve making any choices, i.e. if you give two people the same sequences V_k and y_k, they should be able to follow your instructions and independently produce the same extension of λ.

[2] Fundamental solutions are also often called *Green's functions.*

$LK = \delta$ in the sense of distributions, where $\delta \in \mathscr{D}'(\mathbb{R}^n)$ denotes the Dirac δ-function, $\delta(\varphi) := \varphi(0)$ for $\varphi \in \mathscr{D}(\mathbb{R}^n)$.

(b) Show that if K is a fundamental solution for L, then the linear map

$$\mathscr{D}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) : f \mapsto u, \qquad u(x) := \int_{\mathbb{R}^n} K(x-y) f(y)\, dy$$

associates to every smooth compactly supported function f a smooth solution u to the partial differential equation $Lu = f$. [6pts]

(c) Find a locally integrable function $K : \mathbb{R} \to \mathbb{R}$ that is a fundamental solution for the operator $L := \partial_x^2$, and verify by explicit computation that the prescription in part (b) provides a solution u to $u'' = f$ for any $f \in C_0^\infty(\mathbb{R})$. [6pts]

Problem 4 [43pts total]
Assume X and Y are Banach spaces, and $\mathscr{L}(X, Y)$ denotes the Banach space of bounded linear operators from X to Y, equipped with the operator norm. As usual we will sometimes abbreviate $\mathscr{L}(X) := \mathscr{L}(X, X)$, and the identity map on X will be denoted by $\mathbb{1}_X \in \mathscr{L}(X)$.

(a) Show that an operator $A \in \mathscr{L}(X, Y)$ is injective with closed image if and only if there exists a constant $c > 0$ such that $\|Ax\| \geqslant c\|x\|$ for all $x \in X$. [10pts]

(b) On an infinite-dimensional separable complex Hilbert space $\mathcal{H}$, any choice of orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ determines a linear injection $\Phi : \ell^\infty \hookrightarrow \mathscr{L}(\mathcal{H})$, where ℓ^∞ denotes the Banach space of complex-valued sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ with $\|\mathbf{x}\|_{\ell^\infty} = \sup_{n \in \mathbb{N}} |x_n|$. Concretely, $\Phi(\mathbf{x}) \in \mathscr{L}(\mathcal{H})$ is the unique operator $\mathcal{H} \to \mathcal{H}$ that sends $e_n \mapsto x_n e_n$ for each $n \in \mathbb{N}$. Show that this injection is isometric, i.e. $\|\Phi(\mathbf{x})\| = \|\mathbf{x}\|_{\ell^\infty}$ for all $\mathbf{x} \in \ell^\infty$. Conclude that its image is a closed subspace of $\mathscr{L}(\mathcal{H})$. [5pts]
Remark: Since ℓ^∞ is not separable, the continuous inclusion $\ell^\infty \hookrightarrow \mathscr{L}(\mathcal{H})$ implies that $\mathscr{L}(\mathcal{H})$ also is not separable.

If $A \in \mathscr{L}(X, Y)$ and $B \in \mathscr{L}(Y, X)$ satisfy the relation $BA = \mathbb{1}_X$, then B is said to be a *bounded left-inverse* of A, while A is called a *bounded right-inverse* of B; note that in this situation, A must be injective and B must be surjective.

(c) Prove that an injective operator $A \in \mathscr{L}(X, Y)$ admits a bounded left-inverse if and only if its image is closed and complemented, i.e. $Y = \operatorname{im} A \oplus W$ for some closed subspace $W \subset Y$, where $\operatorname{im} A \subset Y$ is also closed. [10pts]
Hint: Show that if $B \in \mathscr{L}(Y, X)$ is a left-inverse, then $AB \in \mathscr{L}(Y)$ is a projection, and so is $\mathbb{1}_Y - AB$.

(d) Prove that the subset

$$I_0(X, Y) := \left\{ A \in \mathscr{L}(X, Y) \;\middle|\; A \text{ admits a bounded left-inverse} \right\}$$

is open in $\mathscr{L}(X, Y)$. [8pts]

For the last two parts, consider again the injection $\Phi : \ell^\infty \hookrightarrow \mathscr{L}(\mathcal{H})$ from part (b). Prove:

(e) The operator $\Phi(\mathbf{x}) \in \mathscr{L}(\mathcal{H})$ for some $\mathbf{x} = (x_1, x_2, \ldots) \in \ell^\infty$ is injective with closed image if and only if $\inf_{n \in \mathbb{N}} |x_n| > 0$. [5pts]

(f) The subset

$$I_1(\mathcal{H}) := \left\{A \in \mathscr{L}(\mathcal{H}) \;\middle|\; A \text{ is injective with closed image}\right\}$$

is open in $\mathscr{L}(\mathcal{H})$, but the smaller subset

$$I_2(\mathcal{H}) := \left\{A \in \mathscr{L}(\mathcal{H}) \;\middle|\; A \text{ is injective}\right\}$$

is not. [5pts]

Comments: In finite-dimensional vector spaces, all subspaces are closed and complemented, thus all injective linear maps admit bounded left-inverses. Injective operators thus play a very special role in linear algebra, and the fact that small perturbations of injective operators are also injective is often used in applications. The message of this problem is that in infinite-dimensional settings, injectivity on its own is not a sufficiently special condition, but the existence of a bounded left-inverse is.