The symplectic Dehn twist

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The symplectic mapping class group is defined as the space of symplectomorphisms up to symplectic isotopies. In the smooth case, for surfaces, the mapping class group is generated by Dehn twists making them pretty important objects to study. So it is natural to ask whether a symplectic Dehn twist exists and how it would look like. In this talk I present three illustrations of the Dehn symplectic twist to obtain some intuition on the Dehn twist. Then I show that Dehn twist is infinite order in its local model T^*S^2. The talk is based on [Seidel, 1998] and [McDuff & Dusa, 2017].

1 Introduction

Last time we looked at the mapping class group of a surface. Remember that the Mapping class group is defined as the connected isotopy components of the orientation-preserving diffeomorphisms of a surface $MCG(S) = \pi_0(\mathrm{Diff}^+(S))$. We talked about the Thurston-Nielsen-classification which states that every diffeomorphism can be "niceified" by isotopies to a diffeomorphism which is periodic (in the usual sense), reducible or pseudo-Anosov. In particular, we looked at the mapping class group of the torus. If we identify the torus as $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the standard flat metric, then the "niceified" diffeomorphisms are linear maps on the torus and I presented three prototypical examples for the three Thurston-Nielsen classes:

Example 1.1.
1. *$\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is called the Hyperbolic involution and is a prototypical periodic diffeomorphism*
2. *$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is called the Dehn twist and is a prototypical reducible example. It preserves the horizontal curve $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$*
3. *$\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is called Arnold's cat map and is a prototypical pseudo-Anosov. Iterate the linear map on some simple closed curve and it approaches a foliation whose slope is given by the eigenvector of the linear map*

The matrices describe the "nicefied" linear diffeomorphisms. But there is a second way to interpret the matrices (which I used implicitly in the prototypical examples when I called a vector a curve).

The mapping class group acts on the isotopy classes of curves $\pi_1(T^2)$, so it also acts on the Abelianisation $\mathrm{Ab}(\pi_1(T^2)) = H_1(T^2) \cong \pi_1(T^2)$. Note that the last isomorphism holds because the fundamental group of the torus is already Abelian. From this we get a canonical action of the mapping class group on the first homology $MCG(T^2) \circlearrowright H_1(T^2)$ by linear maps. And these agree with the description above! Because isotopies by definition preserve isotopy classes of curves this shows the following:

Lemma 1.2. *The hyperbolic involution, the Dehn twist and Arnold's cat map are not isotopic to the identity.*

Before we start, we give a formal definition of the Dehn twist.

Definition 1.3 (Dehn twist on a cylinder)**.** *Let $\mathbb{R} \times S^1 \subseteq \mathbb{R} \times \mathbb{C}$ be the cylinder. Then the Dehn twist is the map*

$$\tau : \mathbb{R} \times \mathbb{C} \to \mathbb{R} \times \mathbb{C} \tag{1}$$

$$(t, z) \mapsto -e^{\pi i \frac{t}{\sqrt{1+t^2}}} z \tag{2}$$

It has the properties: $\tau(-\infty) = \tau(\infty) = 1, \tau(0) = -1$*.*

Definition 1.4 (Dehn twist on a surface)**.** *Let S be a surface and γ be a simple, closed curve. There is a smooth embedding of a cylinder into a tubular neighbourhood of γ:*

$$\mathbb{R} \times S^1 \hookrightarrow \mathcal{N}(\gamma)$$

We can define a map $\psi_\gamma : S \to S$ that applies a Dehn twist inside $\mathcal{N}(\gamma)$ and is the identity outside.

2 The symplectic mapping class group in dimension 2

Mapping class groups are extremely fun to study. This begs the question, ~~is symplectic geometry fun?~~ does symplectic geometry have a richer theory of mapping class group? i.e., are there space where the symplectic mapping class group is bigger than the smooth mapping class group? i.e., are there elements that are non-isotopic to the identity via symplectomorphisms but isotopic via diffeomorphisms? We start in dimension 2. This is easy and start by we introducing the following tool:

Theorem 2.1 (Orbit-Stabiliser-theorem)**.** *Let $G \circlearrowright X$ be a transitive action. Let $x \in X$. Then the following holds:*

$$X \cong \mathrm{Orbit}_G(x) \cong G/\,\mathrm{Stab}_G(x)$$

as homeomorphisms.

Theorem 2.2. *The symplectic, volume-preserving and smooth mapping class group agree for connected oriented surfaces.*

$$SMCG(S) = VMCG(S) = MCG(S)$$

Proof. We will first show that the smooth MCG agrees with the MCG of volume-preserving diffeos. For $\lambda > 0$ let Ω_λ be the space of volume-forms with $\int_S \omega = \lambda$ for $\omega \in \Omega_\lambda$. The space of orientation preserving diffeomorphisms $\mathrm{Diffeo}^+(S)$ acts on Ω. Note that all forms in Ω_λ belong to the same cohomology class. The stabiliser is given by $\mathrm{Vol}(S)$, the space of volume-preserving diffeomorphisms. The space Ω is convex, so given any two volume-forms $\omega_0, \omega_1 \in \Omega$, we also have $\omega_t = t\omega_0 + (1-t)\omega_1 \in \Omega$. Then $\frac{d}{dt}\omega_t = \omega_0 - \omega_1 = d\sigma$ for some form σ because $[\omega_0] - [\omega_1] = [0]$. By Moser's trick there is smooth isotopy of diffeomorphisms ψ_t^* with $\omega_t = \psi^*\omega_0$ and in particular $\omega_1 = \psi_t^*\omega_0$. This shows that the action of $\mathrm{Diffeo}^+(S)$ on Ω_λ is transitive. By the Orbit-Stabiliser theorem we get $\Omega_\lambda \cong \mathrm{Diffeo}(S)/\mathrm{Vol}(S)$. As Ω_λ is convex, it is contractible and so is $\mathrm{Diffeo}(S)/\mathrm{Vol}(S)$, so in particular the MCGs of smooth maps and volume-preserving smooth maps agree. On surfaces volume forms are exactly the symplectic forms, and the volume preserving diffeos are exactly the symplectomorphisms so their MCGs also agree. □

3 Three illustrations of the symplectic Dehn twist

What about higher dimensions? Can we construct a symplectomorphism which is not isotopic to the identity via symplectomorphisms? We might try to generalize the Dehn twist to the symplectic setting. But first we have to properly define the Dehn twist.

Definition 3.1 (Symplectic Dehn twist)**.** *Examine T^*S^{n-1} with the identification*

$$T^*S^{n-1} = \{q + ip \mid \|q\| = 1, \langle q, p\rangle = 0\} \subseteq \mathbb{C}^n$$

The synplectic Dehn twist for a $(q_0, p_0) \in T^*S^{n-1}$ is the function $\tau(q_0+ip_0) = (q_0 + ip_0)$ defined by the equation

$$\left(q_1 + i\frac{p_1}{|p_1|}\right) = \tau^{|p_0|}\left(q_0 + i\frac{p_0}{|p_0|}\right) := \left(-e^{\pi i \frac{|p_0|}{\sqrt{1+|p_0|^2}}}\left(q_0 + i\frac{p_0}{|p_0|}\right)\right)$$

and by $|p_1| = |p_0|$. So to calculate the value $\tau(q_0+ip_0)$ first scale the imaginary part p_0 to size $\frac{p_0}{|p_0|}$, then plug the complex number in to the formula and finally scale the imaginary part of the result back to $|p_0|$:

$$\mathrm{Re}(\tau(q + ip)) = \mathrm{Re}\left(\tau^{|p_0|}\left(q + i\frac{p}{|p|}\right)\right)$$
$$\mathrm{Im}(\tau(q + pi)) = |p|\mathrm{Im}\left(\tau^{|p_0|}\left(q + i\frac{p}{|p|}\right)\right)$$

This definition agrees with the definition of the Dehn twist in $n = 2$. Also, it behaves similarly in that

- *as $|p_0|$ goes to infinity $\tau(q_0 + ip_0)$ goes to $q_0 + ip_0$*
- *$\tau(q_0 + 0) = -q_0$.*

- *Furthermore, there is some "time" $\tau = |p_0|$ where $-e^{\pi i \frac{\tau}{\sqrt{1+\tau^2}}} = i$.*

Illustration 3.2. *For some $|q| = 1$, the image of the family $\{q + itp \in T^*_q S^{n-1} \mid t \in \mathbb{R}\}$ under the symplectic Dehn twist therefore looks something like in Figure 1*

Note the time $t = \tau$. The multiplication by i exchanges the real/complex or position/momentum components of the T^*S^{n-1}. This makes the symplectic Dehn twist symplectic.

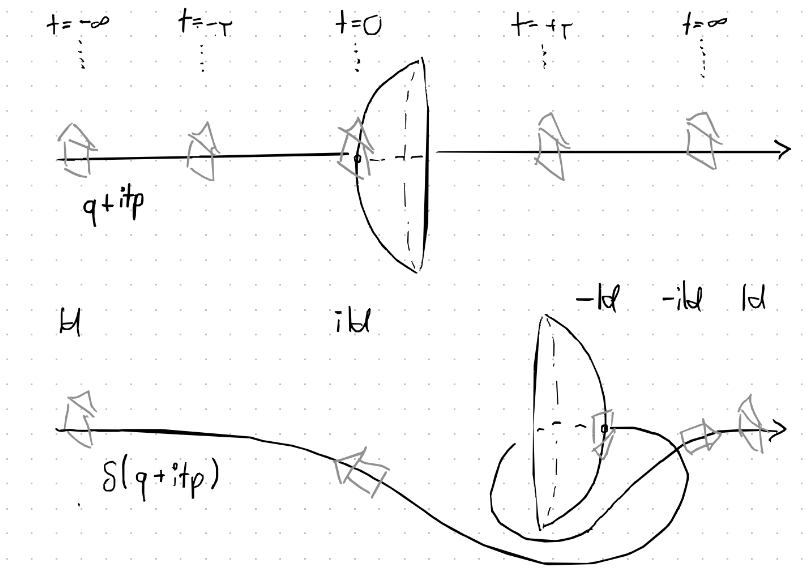

Figure 1: The image of $q + itp \in T^*_q S^{n-1}$ does a twist. (Please replace the τ in the image by t_0)

Illustration 3.3. *The projection of $\{\tau(q + itp) \in T^*_q S^{n-1} \mid t \in \mathbb{R}\}$ to S^2 follows a geodesic. See Figure 2*

Proof. Since we are in $\mathbb{C}^n$, we can identify the cotangent bundle with the tangent bundle $T^*S^{n-1} \cong TS^{n-1}$. Then every point $q + ip \in TS^{n-1}$ can be visualised as a point $p \in S^{n-1}$ with a perpendicular vector attached. Let

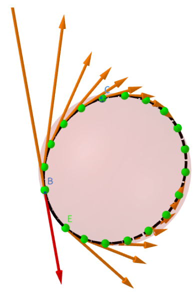

Figure 2: The projection of $\tau(q+itp) \in T_q^*S^{n-1}$ to S^2 follows a geodesic

$|p| = 1$. Then

$$\begin{aligned}
\mathrm{Re}(\tau(q+itp)) &= \mathrm{Re}(\tau^{|t|}(q+ip)) \\
&= \mathrm{Re}\left(e^{\pi i \frac{t}{\sqrt{1+t^2}}}(q+ip)\right) \\
&= \mathrm{Re}(e^{\pi i\cdot s(t)}(q+ip)) \quad \text{for } s(t) \in (-1,1) \\
&= \mathrm{Re}((\cos(\pi s)q - \sin(\pi s)p) + i(\sin(\pi s)q + \cos(\pi s)p)) \\
&= \cos(\pi s)q - \sin(\pi s)p \\
\mathrm{Im}(\tau(q+itp)) &= |t|\mathrm{Im}(\tau^{|t|}(q+ip)) \\
&= |t|(\sin(\pi s)q + \cos(\pi s)p)
\end{aligned}$$

□

Illustration 3.4. *Examine the symplectic singular fibration*

$$\pi : \mathbb{C}^n \to \mathbb{C}, (z_1, \dots, z_n) \mapsto z_1^2 + \cdots + z_n^2$$

It has the fibre

$$F = \pi^{-1}(1) = \{x + iy \in \mathbb{C}^n \mid |x|^2 - |y|^2 = 1, \langle x, y\rangle = 0\}$$

This is symplectically isomorphic to T^*S^{n-1}. Because π is a symplectic fibration, the monodromy of the path $t \mapsto e^{2\pi it}$ is a symplectomorphism on F. This induces the Dehn twist on T^*S^{n-1}.

Corollary 3.4.1. *The symplectic Dehn twist is a symplectomorphism.*

We can use the Dehn twist as follows: Given an arbitrary symplectic manifold M with an embedded Lagrangian sphere L, we can find a symplectic embedding $T^*S^n \hookrightarrow M$ such that S^n maps to L. We can then modify τ to be the identity outside a small neighbourhood of L and obtain a symplectomorphism $\tau_L : M \to M$, the Dehn twist around L.

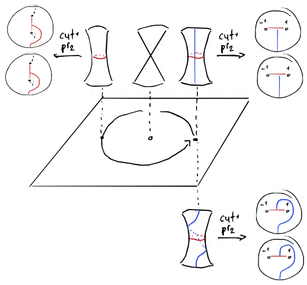

Figure 3: The Dehn twist is obtained from monodromy of the quadratic map.

4 The Dehn twist has infinite order

Theorem 4.1 (Seidel, 1998)**.** *We examine the model space T^*S^2 and its compactly supported symplectomorphisms* $\mathrm{Symp}_c(T^*S^2)$*. Then the Dehn twist τ has the following properties:*

1. *$[\tau_L]$ has infinite order in* $\mathrm{Symp}(T^*S^2)$ *(= the group of compactly supported maps).*

2. *τ_L has order 2 in* $\mathrm{Diffeo}^+(T^*S^2)$*.*

Furthermore, $SMCG(T^*S^2) \cong \mathbb{Z}$.

We note that T^*S^2 is not a closed manifold. It would be better if it were closed. Then we could apply the theory of holomorphic curves to obtain some results. Therefore, we start by compactifying. We embed $T^*S^2 \hookrightarrow S^2 \times S^2$ such that the zero section goes to the anti-diagonal $\bar{\Delta} := \{(x, -x) \in S^2 \times S^2\}$. Then T^*S^2 can be identified as a neighbourhood $N(\bar{\Delta})$ and

$$(S^2 \times S^2) \backslash \iota(T^*S^2) = (S^2 \times S^2) \backslash N(\bar{\Delta}) \cong N(\Delta)$$

where $\Delta = \{(x, x) \in S^2 \times S^2\}$. The compactly supported symplectomorphisms now agree with the symplectomorphisms that restrict to the identity on a neighbourhood of Δ, i.e., $\mathrm{Symp}_c(T^*S^2) = \mathrm{Symp}(S^2 \times S^2, \mathcal{N}\Delta)$. Remember, that we are interested in $SMCG(T^*S^2) = \pi_0(\mathrm{Symp}_c(T^*S^2)) = \pi_0(\mathrm{Symp}(S^2 \times S^2, \mathcal{N}\Delta))$.

By forgetting the fact that a map $\pi_0(\mathrm{Symp}(S^2 \times S^2, \mathcal{N}\Delta))$ is the identity in the neighbourhood of Δ (as opposed to being the identity on just Δ) we get the short exact sequence:

$$1 \to \mathrm{Symp}(S^2 \times S^2, \mathcal{N}\Delta) \hookrightarrow \mathrm{Symp}(S^2 \times S^2, \Delta) \twoheadrightarrow \mathrm{Aut}(\nu\Delta) \to 1$$

where $\mathrm{Aut}(\nu\Delta)$ are the fibre-preserving automorphisms of the normal bundle and they describe in a sense the "difference" of the previous two groups.

This concludes the setup for the proof of theorem 1. Let's import two useful lemmas before we do the actual proof.

Lemma 4.2. *Let $i \in \mathrm{Symp}(S^2 \times S^2, \Delta)$ be the involution that exchanges the two spheres. This acts non-trivially on homology. Furthermore, $\{id, i\}$ and $\mathrm{Symp}(S^2 \times S^2, \Delta)$ are weakly homotopy equivalent.*

Remark: The previous lemma is a black box that uses the theory of holomorphic curves.

Lemma 4.3. $\mathrm{Aut}(\nu\Delta)$*,* $SL_2(\mathbb{Z})$ *and* S^1 *are weakly homotopy equivalent.*

Proof of theorem 1. The short exact sequence above induces a long exact sequence of homotopy groups:

$$... \to \pi_1(\mathrm{Aut}(\nu\Delta))$$
$$\to \pi_0(\mathrm{Symp}(S^2 \times S^2, \mathcal{N}\Delta)) \to \pi_0(\mathrm{Symp}(S^2 \times S^2, \Delta)) \to \pi_0(\mathrm{Aut}(\nu\Delta)) \to 1$$

□

By the previous two lemmas we have $\pi_k(\mathrm{Aut}(\nu\Delta)) = \pi_k(\mathrm{Symp}(S^2 \times S^2, \Delta)) = 0$ for $k > 1$. This implies that $\pi_k(\mathrm{Symp}(S^2 \times S^2, \mathcal{N}\Delta)) = 0$ for $k > 1$. The following short sequence remains:

$$1 \to \underbrace{\pi_1(\mathrm{Aut}(\nu\Delta))}_{\mathbb{Z}} \overset{\partial}{\hookrightarrow} \pi_0(\mathrm{Symp}(S^2 \times S^2, \mathcal{N}\Delta)) \to \underbrace{\pi_0(\mathrm{Symp}(S^2 \times S^2, \Delta))}_{\mathbb{Z}/2} \to 1$$

Let $r_s \in \mathrm{Aut}(\nu\Delta)$ be the map that rotates each plane of $\nu\Delta$ the the angle s. Then the path $r = (r_s)_{0 \le s \le 2\pi}$ generates $\pi_1(\mathrm{Aut}(\nu\Delta))$. The monodromy of r induces a diffeomorphism $\partial(r)$ in $\mathrm{Symp}(S^2 \times S^2, \mathcal{N}\Delta)$ and one can show that this is $[\tau]^2$ (a map that is isotopic to the squared Dehn twist).

Because of the inclusion $\overset{\partial}{\hookrightarrow}$, the mapping class $[\tau]^2$ has infinite order in $\pi_0(\mathrm{Symp}(S^2 \times S^2, \mathcal{N}\Delta))$. As for the isomorphism type of the middle group, because of the s.e.s. it can only be $\mathbb{Z}$ or $\mathbb{Z}/2$. In the first case, r maps to $2 \in \mathbb{Z}$. In the second case, it maps to $(1, 0) \in \mathbb{Z} \times \mathbb{Z}/2$. But since we now that the Dehn twist $[\tau]$ (i.e. the square root of $[\tau]^2$) exists we must have $[\tau]^2 = 2 \in \mathbb{Z}$, as $(1, 0)$ does not have a square root. In total, we have $SMCG(T^*S^2) = \mathbb{Z}$

To prove the second claim we compare the s.e.s above with the following short sequence.

$$\pi_1(\mathrm{Diffeo}(S^2 \times S^2, \Delta)) \to$$
$$\underbrace{\pi_1(\mathrm{Aut}(\nu\Delta))}_{\mathbb{Z}} \overset{\partial'}{\hookrightarrow} \pi_0(\mathrm{Diffeo}(S^2 \times S^2, \mathcal{N}\Delta)) \to \underbrace{\pi_0(\mathrm{Diffeo}(S^2 \times S^2, \Delta))}_{\mathbb{Z}/2} \to 1$$

Two pairs of the terms are isomorphic. To show that $[\tau]^2 = 0$ Seidel constructs an isomorphism $\pi_1(\mathrm{Diffeo}(S^2 \times S^2, \Delta)) \cong \pi_1(\mathrm{Aut}(\nu\Delta))$.