Exotic Symplectic Structures

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1 Abstract

We all know the symplectic structure on $\mathbb{R}^{2n}$. But are there other "exotic" structures, not symplectomorphic to the standard one? Answering this seemingly simple question requires a dive into the theory of J-holomorphic curves. We explain key ideas of this theory in a way that can be understood by those unfamiliar with the topic. We then apply these ideas to the problem of exotic $\mathbb{R}^{2n}$.

2 Introduction

On $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ as a vector space we have the standard symplectic structure and it is easy to see that this is unique up to linear symplectomorphism. Through identifying the tangent spaces of $\mathbb{C}^n$ with $\mathbb{C}^n$ this also gives rise to a symplectic structure on $\mathbb{C}^n$ as a manifold. But is this structure unique up to symplectomorphism? Why care about it?

If it would be unique this could open up arguments going like "construct a symplectic manifold, prove that it is diffeomorphic to $\mathbb{C}^n$, then it is symplectomorphic". That would sure be nice! Would be even nice if we could prove such theorems for all kinds of symplectic manifolds, not just $\mathbb{C}^n$. But $\mathbb{C}^n$ seems like a good start.

Now the astute reader might have noticed my repeated usage of words such as "would". Sadly symplectic structures on $\mathbb{C}^n$ are not unique. It will be our goal to construct such a non-standard, i.e. exotic, symplectic structure. The general idea of the construction can be understood quite easily, but it uses some heavy machinery from the theory of J-holomorphic curves and the h-principle. We will explore the former, while blackboxing the latter.

Most of what we say can be found in [MS12]. The construction of the exotic symplectic $\mathbb{C}^n$ is due to [ALP94].

3 Proof Idea

Before getting into any details, one has to mention the special case $n = 1$:

Theorem 3.1. *There are no exotic symplectic structures on* $\mathbb{C}$

Proof. The important idea is realizing that every symplectic form on $\mathbb{C}$ is a volume form. Having two forms ω_0, ω_1 one can define $\omega_t = (1-t)\omega_0 + t\omega_1$. We can assume WLOG that ω_0, ω_1 have the same orientation, then ω_t is a volume form for every t and by Moser's theorem ω_0 and ω_1 are symplectomorphic. □

Now onto the general case. As always with exotic structures, one first finds a property that the standard structure has and then construct one without this property. In our case this is the following theorem

Definition 3.2. *Let* $\lambda \in \Omega^1(\mathbb{C}^n)$ *s.t.* $\mathrm{d}\lambda = {\omega_0}$[1]. *A Lagrangian submanifold* $L \subset \mathbb{C}^n$ *is called exact if* $\lambda_{|L}$ *is exact.*

Note that both existence of λ and the independence of exactness from the choice of λ depend on the fact that $\mathbb{C}^n$ has trivial first cohomology.

Exact Lagrangians are interesting, because of the following:

Lemma 3.3. *The standard symplectic structure* ω_0 *on* $\mathbb{C}^n$ *does not admit any closed exact Lagrangians.*

This Lemma is going to be proven later, but for now we take it as a black box and try to use it to construct an exotic structure.

Now onto construction of the exotic symplectic structure. We are going to construct a smooth immersion $F : \mathbb{C}^n \to \mathbb{C}^n$ that sends a given closed Lagrangian submanifold L of $(\mathbb{C}^n, \omega_0)$ onto an immersed exact Lagrangian submanifold W of $(\mathbb{C}^n, \omega_0)$. The pullback $F^*\omega_0$ is then a symplectic structure (because F is an immersion) on $\mathbb{C}^n$ in which L is a closed exact Lagrangian submanifold, which implies that it can't be symplectomorphic to the standard symplectic structure i.e. is exotic.

To construct such an immersion F, we take $\mathbb{T}^n = (S^1)^n \subseteq \mathbb{C}^n$ as our closed Lagrangian. This is a Lagrangian as the n-fold product of a Lagrangian, were S^1 is Lagrangian since alternating forms necessarily vanish on every 1-dimensional subspace. To turn this into an exact Lagrangian immersion, let $f_t : S^1 \to \mathbb{C}$ be a regular homotopy from the standard inclusion to one with 2 double double points. The left and right side of such an immersion have the same orientation as each other, and the opposite orientation as the middle part, allowing us to pick the proportion in such a way that the integral of ω_0 over the interior is zero. Then for any primitive λ of ω_0 :

$$\int_{S^1} f_1^*\lambda = \int_{f_1(S^1)} \lambda = \int_A \omega_0 = 0$$

where A is the area enclosed by $f_1(S^1)$.[2]. A 1-form is exact iff the integral of all possible pullbacks to S^1 are exact. Since integrals of other pullbacks are just

[1]Such a λ always exists since ω is closed and $\mathbb{C}^n$ has trivial cohomology

[2]Of course A isn't really a smooth manifold, but except for the middle point it can be split up into two smooth manifolds with corners, which suffices for our intents

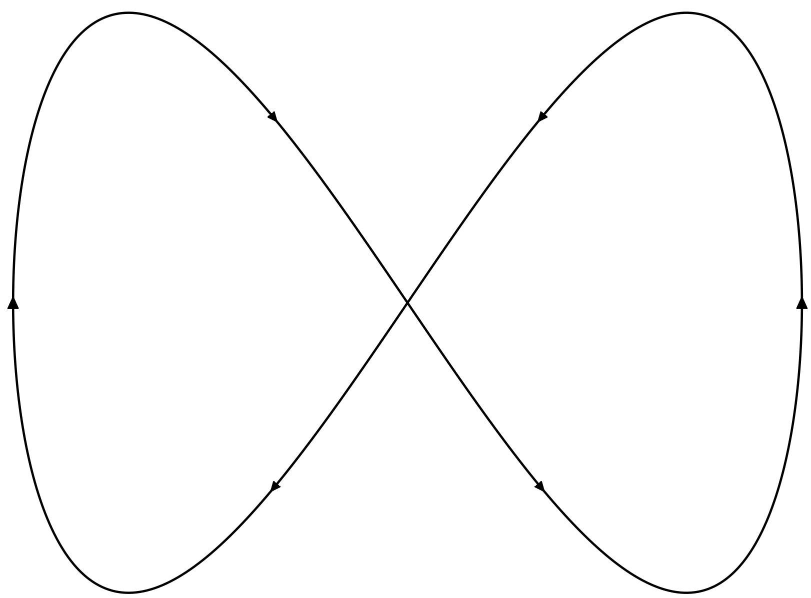


Figure 1: An immersed exact Lagrangian

integer multiples of this integral, this shows that $\lambda_{|S^1}$ is exact. $f_1(S^1)$ is also Lagrangian since it is still something 1-dimensional

Defining $F_t : \mathbb{T}^n \to \mathbb{C}^n$ as the n-fold product of f_t now gives a regular homotopy between the inclusion of $\mathbb{T}^n$ and the immersion of an exact Lagrangian, because the canonical symplectic form on $\mathbb{C}^n$ is the n-fold sum of the symplectic forms on all the $\mathbb{C}$'s, hence $\lambda_0 = \sum_{i=1}^n \lambda_i$ is a primitive of $\omega_0 = \sum_{i=1}^n \omega_i$ and in particular is exact on $F_1(\mathbb{T}^n)$ since all the λ_i are exact on various parts of the product.

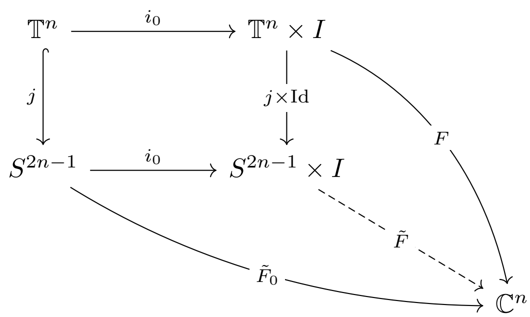
It remains to extend F_1 to $\mathbb{C}^n$. To accomplish this, note that one can identify:

$$\mathbb{T}^n = (S^1)^n = \{ (z_1, \ldots, z_n) \subseteq \mathbb{C}^n \mid |z_1| = |z_2| = \cdots = |z_n| = 1 \}$$

In this identification all elements of $\mathbb{T}^n$ have norm $\sqrt{n}$ and can thus (with proper scaling) be seen as a subset of

$$S^{2n-1} = \{ z \in \mathbb{C}^n \mid |z| = 1 \}$$

The Smale-Hirsch h-principle for immersions (which we are going to take as a black box, for a reference see chapter 8 of [CEM24]) now tells us, that if we can extend $F_0 : \mathbb{T}^n \to \mathbb{C}^n$ to an immersion $\tilde{F}_0 : S^{2n-1} \to \mathbb{C}^n$, then we can extend our whole regular homotopy of immersions to $\tilde{F}_t : S^{2n-1} \to \mathbb{C}^n$. Or expressed in diagrams, if we can find a $\tilde{F}_0$ to make the following diagram commute, then the existence of $\tilde{F}_t$ is guaranteed.



Properly identifying $\mathbb{T}^n$ as a subset of S^{2n-1} like above directly provides us with such a $\tilde{F}_0$ and thus $\tilde{F}_t$.

In particular we get $\tilde{F}_1 : S^{2n-1} \to \mathbb{C}^n$ with $(\tilde{F}_1)_{|\mathbb{T}^n} = F_1$, which is all we require. Since $\mathbb{T}^n$ is compact, one can find two smoothly embedded disc $\mathbb{T}^n \subset \mathbb{D}_1^{2n-1} \subsetneq \mathbb{D}_2^{2n-1} \subset S^{2n-1}$. Taking $\mathbb{D}_2^{2n-1}$ as a submanifold of $\mathbb{C}^n$, find a tubular neighborhood $\mathcal{U}$ of the disc **in** $\mathbb{C}^n$ and, using an immersed tubular neighborhood of $\tilde{F}_1(\mathbb{D}_2^{2n-1})$, extend $(\tilde{F}_1)_{|\mathbb{D}_2^{2n-1}}$ to an immersion $G : \mathcal{U} \to \mathbb{C}^n$.[3] Concatenation with an immersion $\varphi : \mathbb{C}^n \to \mathcal{U}$ that is constant on $\mathbb{D}_1^{2n-1}$ then gives the immersion $F \coloneqq G \circ \varphi : \mathbb{C}^n \to \mathbb{C}^n$ which maps $\mathbb{T}^n \subseteq \mathbb{C}^n$ onto the immersed exact Lagrangian $F_1(\mathbb{T}^n)$.

4 J-holomorphic curves

Proving the existence of exotic symplectic structures heavily relies on the machinery of J-holomorphic (or pseudo-holomorphic) curves, a generalization of holomorphic functions to the context of symplectic manifolds.

Definition 4.1. *Given a smooth manifold M, an almost complex structure J is a bundle morphism $J : TM \to TM$ such that $J^2 = -1$. We call (M, J) an almost complex manifold.*

This can be seen as a direct generalization of "multiplication by i".

To define J-holomorphic functions in the general case, let us recall the classical case: A smooth function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if and only it fulfills the Cauchy-Riemann Equations i.e. if for $f(x + iy) = u(x, y) + iv(x, y)$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

[3] A tubular neighborhood gives us a diffeomorphism $j : \mathcal{U} \to N(\mathbb{D}_2^{2n-1})$, where $N(\mathbb{D}_2^{2n-1})$ denotes the normal bundle. The immersion $(\tilde{F}_1)_{|\mathbb{D}_2^{2n-1}}$ maps this normal bundle to the normal bundle of the image, which we can identify via an immersion with an open subset of $\mathbb{C}^n$

Fulfilling these equations is equivalent to fulfilling the matrix equation:

$$\begin{aligned} &\begin{pmatrix} \partial_y u & -\partial_x u \\ \partial_y v & -\partial_x v \end{pmatrix} = \begin{pmatrix} -\partial_x v & -\partial_y v \\ \partial_x u & \partial_y u \end{pmatrix} \\ \iff \quad &\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \end{aligned}$$

Note that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =: J$ maps the x coordinate to y and the y coordinate inverted to x i.e. is equal to multiplication by i when identifying $\mathbb{C}$ with $\mathbb{R}^2$, allowing us to express this equation succinctly as

$$Df \cdot i = i \cdot Df$$

So in other words: A smooth function is holomorphic iff its differential is complex linear.

This idea can now be generalized to arbitrary almost-complex manifolds:

Definition 4.2. *Let (Σ, j) be a Riemannian surface and (M, J) be almost-complex manifolds. A map $u : \Sigma \to M$ is called J-holomorphic (or pseudoholomorphic) if*

$$Tu \circ j = J \circ Tu$$

Sometimes it is convenient to word this condition slightly different by splitting the map Tu up into a complex linear and complex anti-linear part:

$$Tu = \frac{1}{2} \underbrace{(Tu + J \circ Tu \circ j)}_{\text{Complex Anti-Linear}} + \overbrace{\frac{1}{2}(Tu - J \circ Tu \circ j)}^{\text{Complex Linear}}$$

Calling the complex anti-linear part $\bar{\partial}_J(u)$, a map u is J-holomorphic if and only if $\bar{\partial}_J(u) = 0$.

At last, we can also express this condition in local coordinates. Σ is a Riemannian surface, allowing us to pick an atlas with holomorphic coordinate changes (also called conformal coordinates). Given a local representation $u_\alpha : \mathbb{C} \supset \mathcal{U}_\alpha \to M$ with coordinates $z = s + it$ on Σ, we can write:

$$\begin{aligned} \bar{\partial}_J(u_\alpha) &= \frac{1}{2}(Tu_\alpha + J \circ Tu_\alpha \circ i) \\ &= \frac{1}{2}(\partial_s u_\alpha \, \mathrm{d}s + \partial_t u_\alpha \, \mathrm{d}t - J(u_\alpha) \circ \partial_s u_\alpha \, \mathrm{d}t + J(u_\alpha) \circ \partial_t u_\alpha \, \mathrm{d}s) \\ &= \frac{1}{2}(\partial_s u_\alpha + J(u_\alpha) \circ \partial_t u_\alpha) \, \mathrm{d}s + \frac{1}{2}(\partial_t u_\alpha - J(u_\alpha) \circ \partial_s u_\alpha) \, \mathrm{d}t \end{aligned}$$

This gives us two term that have to become zero, but applying $J(u_\alpha)$ to the $\mathrm{d}t$ term turns it into the $\mathrm{d}s$ term, giving us just one equation:

$$\partial_s u_\alpha + J(u_\alpha) \partial_t u_\alpha = 0$$

While almost complex manifolds form the minimal setting required to define J-holomorphic curves, they only really start to thrive in connection with symplectic geometry.

Definition 4.3. *Given a symplectic manifold (M, ω), an almost complex structure J on M is called ω-tame if*

$$\omega(v, Jv) > 0$$

for all non-zero $v \in TM$. We denote the set of all almost complex structures tamed by ω as $\mathcal{J}_\tau(M, \omega)$

This condition is point-wise open, turning $\mathcal{J}_\tau(M, \omega)$ into an open subset of $\mathcal{J}(M)$ (the set of all almost complex structures on M). This is usually sufficient, but a stronger condition is often more convenient to work with:

Definition 4.4. *Given a symplectic manifold (M, ω), an almost complex structure J on M is called ω-compatible if it is ω-tame and*

$$\omega(Jv, Jw) = \omega(v, w)$$

We denote the space of ω-compatible almost complex structures as $\mathcal{J}(M, \omega)$.

A lot of results in this talk could also be stated for ω-tame almost complex structures, but we are going to limit ourselves to ω-compatible ones, for the sake of both time and clarity.

Combining a symplectic form with a compatible almost complex structure gives us even more structure:

Definition 4.5. *Let (M, ω) be a symplectic manifold with ω-compatible almost complex structure J. Then we define the metric[4] induced by J as*

$$\langle v, w \rangle_J \coloneqq \omega(v, Jw)$$

Note that this metric is invariant with respect to J, i.e.

$$\langle v, w \rangle_J = \omega(v, Jw) = \omega(Jv, -w) = \langle Jv, Jw \rangle_J$$

Having such a metric allows us to define the energy of a curve as earlier:

Definition 4.6. *Let (M, ω) be a symplectic manifold with compatible almost complex structure J and $u : \Sigma \to M$ a smooth map. The* ***energy*** *of u is defined as*

$$E(u) \coloneqq \frac{1}{2} \int_\Sigma |Tu|_J^2 \, \mathrm{dvol}_\Sigma$$

where we define the norm of Tu at a point $z \in \Sigma$ as

$$|Tu|_J \coloneqq |\xi|^{-1} \sqrt{|Tu(\xi)|_J^2 + |Tu(j_\Sigma \xi)|_J^2}$$

for $0 \neq \xi \in T_z\Sigma$.

[4]This is a symmetric because $\omega(v, Jw) = -\omega(Jw, v) = \omega(Jw, JJv) = \omega(w, Jv)$ and non-degenerate because ω is non-degenerate and J a bijection.

While energy can be made sense of with just a Riemannian metric, having a metric induced by a symplectic structure and a compatible almost complex structure provides us with an alternate definition

Lemma 4.7 (Energy Identity)**.** *Let (M, ω) be a symplectic manifold with compatible almost complex structure J and $u : \Sigma \to M$ a smooth map. Then the energy of u is equal to:*

$$E(u) = \int_\Sigma |\bar{\partial}_J(u)|_J^2 \, \mathrm{dvol}_\Sigma + \int_\Sigma u^*\omega$$

In particular if u is J-holomorphic:

$$E(u) = \int_\Sigma u^*\omega \, \mathrm{dvol}_\Sigma$$

Remark 4.8. *A direct consequence of the energy identity is that the energy of a J-holomorphic curve depends only on it's homology class i.e. is a purely topological property. This isn't true for general almost complex structures and metrics, but requires the additional symplectic structure. This also allows us to interpret J-holomorphic curves as "minimal surfaces", since they minimize the energy in their homology class.*

Proof. Since Σ is a Riemannian surface we can locally choose conformal coordinates $z = s + it$ on $\mathcal{U} \subset \Sigma$, giving us:

$$\begin{aligned}
\frac{1}{2}|Tu|_J^2 &= \frac{1}{2}\left(|\partial_s u|_J^2 + |\partial_t u|_J^2\right) \\
&= \frac{1}{2}\left(\langle \partial_s u, \partial_s u\rangle_J + \langle \partial_t u, \partial_t u\rangle_J\right) \\
&= \frac{1}{2}(\langle \partial_s u + J\partial_t u - J\partial_t u, \partial_s u\rangle_J + \langle J\partial_t u + \partial_s u - \partial_s u, J\partial_t u\rangle_J \\
&= \frac{1}{2}\left|\partial_s u + J\partial_t u\right|_J^2 - \langle \partial_s u, J\partial_t u\rangle_J \\
&= |\bar{\partial}_J(u)|_J^2 + \omega(\partial_s u, \partial_t u)
\end{aligned}$$

□

J-holomorphic curves are quite abstract, but having this characterization in terms of "minimal surfaces" can already be quite helpful and give intuition why it is "hard" for a curve to be J-holomorphic.

4.1 Bubbling

J-holomorphic curves, like symplectic geometry itself, are powerful, because they lie on a sweet spot between Algebra, Analysis and Geometry. Having already talked about some algebraic and geometric aspects of them, let us now look at the analytic aspects of them. Given a sequence of J-holomorphic curves, can we expect a subsequence to converge to another J-holomorphic curve?

Theorem 4.9. *Let (M, J) be a compact almost complex manifold, $L \subseteq M$ a compact totally real submanifold and J_ν a sequence of almost complex structures on M that converges in the C^∞-topology to J. Moreover let $(\Sigma, j_\Sigma, \mathrm{dvol}_\Sigma)$ be a compact Riemann surface and $u_\nu : (\Sigma, \partial\Sigma) \to (M, L)$ a sequence of J_ν-holomorphic curves such that*

$$\sup_\nu \|du_\nu\|_{L^\infty} < \infty.$$

Then after passing to a subsequence u_ν converges to a J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, L)$ in the C^∞ topology. □

A slightly more general version of this theorem and a proof of it can be found in [MS12] theorem 4.1.1.

This result is really nice, but often not enough. As we saw above, the energy of J-holomorphic curves in a given homology class is constant, but this energy bound sadly does not imply a bound of the differentials as required for theorem 4.9. But almost as much is true, giving rise to the following theorem

Theorem 4.10 (Convergence modulo bubbling)**.** *Let $J^\nu \in J_\tau(M, \omega)$ be a sequence of ω-tame almost complex structures converging to J in the C^∞-topology and $L \subseteq M$ a compact Lagrangian. Moreover let $(\Sigma, j_\Sigma, \mathrm{dvol}_\Sigma)$ be a compact Riemann surface and $u_\nu : (\Sigma, \partial\Sigma) \to (M, L)$ a sequence of J_ν-holomorphic curves such that*

this should stay on the board

$$\sup_\nu E(u_\nu) < \infty.$$

Then after passing to a subsequence there exists a J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, L)$ and a finite set $Z = \{z_1, ..., z_l\} \subset \Sigma$ such that the following holds

1. *u_ν converges to u in $C^\infty_{loc}(\Sigma \setminus Z, M)$, i.e. on compact subsets not containing any points of Z.*
2. *At each z_i at least one non-constant sphere or disk bubbles off.*
3. *The energy of u plus the energy of the spheres and disks bubbling off is precisely equal to $\lim_{\nu\to\infty} E(u_\nu)$.*

Again a more general version of this and a proof can be found in [MS12] theorem 4.6.1.

A sphere or disk bubbling off is best explained through a picture (as given in the talk).

One can show that for a given symplectic manifold there is a positive lower bound of the energy of J-holomorphic bubbles, usually denoted by $\hbar$. This is the reason why only finitely many bubbles can arise.

In general bubbling will occur. But in some cases there might be further assumptions preventing the existence of bubbles. In such cases the set Z above is empty and $u_\nu \to u$ converges everywhere again, just as in theorem 4.9. This idea will be used in our proof of theorem 5.1.

5 The actual proof

To proof the non-existence of exact Lagrangians we first have to reformulate the statement into the language of J-holomorphic curves. The underlying theorem is actually:

Theorem 5.1. *Let $L \subset (\mathbb{C}^n, \omega_0)$ be a closed Lagrangian. Then there exists a non-constant i-holomorphic curve $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^n, L)$.*

Using the energy identity, the non-existence of exact Lagrangians immediately follows from this since non-constant curves have non-zero energy:

$$0 < E(u) = \int_{\mathbb{D}} u^*\omega_0 = \int_{\partial\mathbb{D}} u^*\lambda$$

(with λ again being some primitive of ω_0). If $\lambda_{|L}$ would be exact this integral would be zero, thus it can't be exact, thus exact closed Lagrangians can not exist.

Now "only" the proof of the actual theorem remains. A complete proof would take up too much time/space, so we will try to give the important ideas and refer to [MS12] for the more technical details.

One conceptional problem of the statement is that the techniques we developed so far have not given us any way to prove existence of J-holomorphic curves, we only talked about the properties they have to fullfill. To deal with this, we are going to do a proof by contradiction and assume that there are no non-constant holomorphic curves $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^n, L)$. This might seem like a step backwards at first (since we now don't have any holomorphic curves to study anymore), but this allows us to use a quite usefull techniques: The theory of perturbed J-holomorphic curves. Recall that J-holomorphic curves are solutions of the following equation:

$$\partial_t u + i\partial_s u = 0$$

and by assumption only constant curves are solutions to this, but what if we perturb this equation as follows:

$$\partial_t u + i\partial_s u = \lambda a$$

for some $\lambda \in [0, 1], a \in \mathbb{C}^n$. For some given $a \in \mathbb{C}^n$ we are going to define the set of perturbed solutions:

$$\mathcal{M}(a) := \{(u, \lambda) \mid u\colon (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^n, L) \text{ s.t. } \partial_t u + i\partial_s u = \lambda a, [u] = 0\}$$

Here $[u] = 0$ means that the relative homology class of u should vanish. Note that by assumption for $\lambda = 0$ all curves are constant and can thus be identified with L itself. One could hope for this not just being a set, but actually a $n+1$ dimensional manifold that is compact and has boundary $\partial\mathcal{M}(a) = \{\, (u, \lambda) \in M(a) \mid \lambda \in \{0, 1\}\,\}$. But at the same time we can show that for a with $|a|$

large enough this equation is not solvable for $\lambda = 1$ (the intuition here being that u would move around "too much" and thus leave L). This is not yet a contradiction, but using the evaluation map $\mathrm{ev} : \mathcal{M} \to L$ that maps (u, λ) to $u(1, 0)$ we get for some regular value $z_0 \in L$ that $\mathrm{ev}^{-1}(\{z_0\})$ is a compact 1-dimensional manifold with only one boundary point (the constant map to z_0 and 0), but this can't be by the classification of compact 1-manifolds!

To complete the proof we now of course have to show that all of these properties actually hold. Sadly proving that this is actually a $n+1$-dimensional manifold is quite hard[5]: We need to realize this it as a zero section of some infinite dimensional Banach[6] manifold and use some rather complicated analytical trickery to make everything work, which is why we leave the details of this to [MS12]. The good news are that one at least gets the boundary property in the same statement, so skipping this step leaves us only with proving that $\mathcal{M}(a)$ is compact, for which we can use our bubbling techniques.

Before we can apply bubbling, we first have to fix a minor problem: The curves we are dealing with are (mostly) not J-holomorphic, since they only fulfill a perturbed version of the J-holomorphic equation. But this can be fixed by using a trick by Gromov which replaces $\mathbb{C}^n$ by $\mathbb{D} \times \mathbb{C}^n$, L by $\partial\mathbb{D} \times L$ and every curve by it's graph i.e. $\tilde{u} : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{D} \times \mathbb{C}^n, \partial\mathbb{D} \times L), \tilde{u}(z) = (z, u(z))$. For every perturbation λa there is then an almost-complex structure $\mathcal{J}_\lambda$ such that a λa perturbed holomorphic curve has a $\mathcal{J}_\lambda$-holomorphic graph. Define:

$$\mathcal{J}_\lambda \coloneqq \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -i\lambda a & -\lambda a & i \end{pmatrix}$$

This is an almost-complex structure with exactly the required properties. Now any sequence $(u^\nu, \lambda^\nu) \subseteq \mathcal{M}(a)$ can be turned into one of $\mathcal{J}_{\lambda^\nu}$-holomorphic curves $(\tilde{u}^\nu)$. Since $\lambda \in [0,1]$ we can assume w.l.o.g. that $\lambda^\nu \to \lambda^*$ and thus $\mathcal{J}_{\lambda^\nu} \to \mathcal{J}_{\lambda^*} =: \mathcal{J}_*$. To apply our bubbling theorem we have to show that the energy of our new curves is still bounded, which is a small calculation:

$$\begin{aligned} E(\tilde{u}) &= \int_{\mathbb{D}} |\partial_s \tilde{u}|^2 \,\mathrm{d}s \wedge \mathrm{d}t \\ &= \pi + \int_{\mathbb{D}} \langle \partial_s u, -i\partial_t u + \lambda a \rangle \,\mathrm{d}s \wedge \mathrm{d}t \\ &= \pi + \int_{\mathbb{D}} \omega_0(\partial_s u, \partial_t u) \,\mathrm{d}s \wedge \mathrm{d}t + \int_{\mathbb{D}} \partial_s(\lambda\langle a, u\rangle) \,\mathrm{d}s \wedge \mathrm{d}t \\ &\quad - \int_{\mathbb{D}} (\partial_s \lambda \langle a, z\rangle) \circ u \,\mathrm{d}s \wedge \mathrm{d}t \\ &= \pi + 0 + \int_0^{2\pi} \cos\theta \lambda \langle a, u(e^{i\theta})\rangle \,\mathrm{d}\theta - \int_{\mathbb{D}} (\partial_s \lambda \langle a, z\rangle) \circ u \,\mathrm{d}s \wedge \mathrm{d}t \end{aligned}$$

[5]starting with the fact that this might not even be a manifold at all, we might have to perturb our curves ever so slightly, but we promise that all other assumptions and statements still work under small enough perturbations.

[6]Something that makes this even more annoying is the fact that smooth and Banach usually don't mix, forcing us to use Sobolev completions to work with weakly differentiable functions.

which gives us boundedness because of compactness. Our bubbling theorem also requires compactness of the co-domain . With these prerequisites fulfilled, bubbling can finally be applied and we can assume w.l.o.g. that $\tilde{u}^\nu$ itself converges to some $\tilde{u}^*$ modulo bubbling at the points $z_1, \dots, z_\ell$. So if we could somehow get rid of these bubbles we would have shown compactness. To do that we have to take a closer look at a bubble e.g. the one at z_1. Rigorously, a bubble at z_1 implies that there is a sequence of conformal rescaling r^ν that shrink $\mathbb{D}$ to smaller and smaller neighborhoods of z_1 and such that $\tilde{u}^\nu \circ r^\nu$ converges to a $\mathcal{J}_*$-holomorphic sphere/disc. But since the first component of $\tilde{u}^\nu$ is the identity for every ν and r^ν shrinks smaller and smaller, the first component of the limit has to be constant! But looking at $\mathcal{J}_\lambda$ one realizes that $\mathcal{J}_\lambda$-holomorphic curves with constant first component are i-holomorphic curves in the second component and these cannot exist by the assumption. Thus bubbling cannot occur and $\tilde{u}^\nu$ actually converges to some $\tilde{u}^*$. Since all $\tilde{u}^\nu$ are the identity in the first component $\tilde{u}^*$ is also the identity in the first component and thus induces a u^* with $(u^\nu, \lambda^\nu) \to (u^*, \lambda^*)$ and thus $\mathcal{M}(a)$ is compact.

how to fix this?

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