Exotic finite type Stein domains

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06/02/2026

1 Introduction

Stein manifolds can be thought of as complex affine schemes, while Weinstein manifolds as those (non-closed) symplectic manifolds built up using a symplectic Morse theory. The former very rigid and the latter quite flexible, though not as much as usual smooth Morse theory. For instance, this is exemplified by the failure of a J-convex h-cobordism theorem. Due to Cieliebak-Eliashberg we know that, up to deformation, Stein structures can be dealt with as Weinstein structures, so with an analog of Morse theory. The question of "fake" or "exotic" finite type Stein $\mathbb{C}^N$'s can be then brought to the more flexible world of symplectic geometry, where exotic symplectic vector spaces were known to exist. Detecting those exotic examples comes down to a symplectic tool that is known to be sensitive to rigid properties of symplectic manifolds while insensitive to flexible ones. This is symplectic homology, a Hamiltonian Floer theory for non-compact manifolds tailored to completions. The goal of this talk is to elaborate on these topics and explain some elements of McLean's result on exotic $\mathbb{C}^N$'s.

1.1 Recall of definitions and facts

- Recall that a **Stein manifold** is a properly embedded complex submanifold of $\mathbb{C}^N$. Affine (complex) algebraic varieties are examples (but not all of them, e.g. open Riemann surfaces). The radial function on $\mathbb{C}^N$ is exhausting (proper and bounded from below) and i-convex (aka strictly plurisubharmonic). Grauert's theorem asserts that a complex manifold (V, J) which admits an exhausting J-convex function is Stein, and so we can take that as an alternative characterization. By definition of J-convexity, $\omega_\phi := -dd^{\mathbb{C}}\phi$, where $d^{\mathbb{C}}\phi = d\phi \circ J$, is a non-degenerate and compatible with J. This makes Stein manifolds symplectic (and Kähler). The corresponding gradient vector field of ϕ is Liouville (expands ω_ϕ along its flow lines). Hence, Stein manifolds are examples of **Weinstein manifolds** (exact symplectic form + exhausting generalized Morse function + complete Liouville gradient like vector field), which are naturally a lot more flexible (no integrability of complex structure, for example). In light of this, the main theorem of the seminar is quite mindbreaking: *the map from Stein to Weinstein is a weak homotopy equivalence.*[1]

- In this, it is important to note that this reduces the classification of Stein manifolds *up to deformation*, not biholomorphism, to that of Weinstein manifolds up to deformation. Here is a great consequence of this: *an almost complex manifold of complex dimension n (different from 2!) that admits an exhausting Morse function with critical points of index at most n can be made Stein after homotoping the almost complex structure.* Still the question of how much flexibility is held within the Weinstein world is left for the next section.

- A **Liouville domain** is an exact symplectic manifold with contact type boundary. These admit completions: exact symplectic manifolds obtained by attaching an infinite cone in a

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[1] Actually, this is conjectural. What we show is a slighly weaker statement about Stein and Weinstein structures with a fixed generalized Morse function and a certain path-lifting property for the mpre general map.

precise way. An **Liouville isomorphism** of domains is a diffeomorphism of their completions that is exact symplectomorphic, i.e. the difference between the domain Lioville 1-form and the pulled one from the target is exact (i.e. the differential of a compactly supported function). This version of isomorphism only pays attention to the *qualitative* symplectic geometry of the domain and not the quantitative. A version of Moser's trick implies that deformation of Liouville structures are Liouville isomorphic (**Liouville stability**). Also, a symplectomorphism of Liouville domains is diffeotopic to an exact one. More generally, a Liouville manifold is an exact symplectic manifold such that the vector field dual to the Liouville form is complete and there is an exhaustion of compact set with smooth boundaries along which this vector field points outwards from. Such an object is called of **finite type** if its skeleton (flowing to $-\infty$ all those compact sets) is compact and this is equivalent to it being the completion of a Liouville domain. Recall that the skeleton is isotropic. We can hence speak of a finite type Stein manifold, as it is a Liouville manifold. This notion for Stein is equivalent to admitting an exhausting J-convex function with only finitely many critical points.

- A Lagrangian in a symplectic manifold is a submanifold of maximal dimension for which the symplectic form vanishes. In the exact case, the primitive of the symplectic form defines a 1-form on the Lagrangian with vanishing derivative, hence Lagrangians in exact symplectic manifolds have a special first cohomology class associated to them. A natural class of Lagrangians is then the **exact Lagrangians**, those for which the Liouville 1-form on them is exact. Exact symplectomorphism preserve exact Lagrangians.

1.2 The main question

- Are there finite type Stein manifolds that are not Stein equivalent but diffeomorphic to $\mathbb{C}^n$? As mentioned before, in light of the Cieliebk-Eliashberg theorem we can bring this into Morse-theoretic grounds.

Theorem. *Contractible finite type Stein manifolds in complex dimension* $n > 2$ *are diffeomorphic to* $\mathbb{C}^n$.

Proof. Let (J, ϕ) be the Stein structure on M in question, we can assume ϕ is Morse and for R large enough M_R is diffeomorphic to M (finite type). One uses the h-cobordism theorem to deduce the result after showing that ∂M_R is simply connected. This follows by considering $\psi := R - \phi$ a Morse function with critical points of index at least n as ϕ has critical points of index at most n. From the fact that $n > 2$ we see no fundamental group can be created with the handle attachments. □

- This, however, cannot be pushed further carelessly. The result of McLean (following Seidel-Smith and completed by Abouzaid-Seidel) will imply that *there exists a Stein cobordism* (W, J) *diffeomorphic to* $\mathbb{S}^{2n-1} \times [0,1]$ *for which the corresponding* J*-convex function* ϕ *cannot be chosen without critical points.* That being said, there is a class of *flexible* Weinstein manifolds (e.g. subcritical ones)[2] for which the h-cobordism theorem holds (after using the Weinstein-Stein road):

Theorem. *Any flexible Stein structure* (J, ϕ) *on a product cobordism* $M \times [0,1]$ *of dimension* $2n > 4$ *admits a* J*-convex function without critical points.*

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[2] More generally, these are the ones that can be critical, but the critical handles may only be attached along *loose* Legendrians. In complex dimension two, these are only the subcritical ones.

So, if we are to find exotic examples, one has to avoid flexible structures. An interesting thing to note is that stabilizing $V \times \mathbb{C}$ a Stein manifold (V, J, ϕ) makes it flexible (makes it subcritical) and so the theorem above implies that if V has diffeomorphic to $\mathbb{C}^n$ but not Stein-equivalent, $V \times \mathbb{C}$ would be indeed equivalent to $\mathbb{C}^{n+1}$.

- An indication this may be possible can be found in a theorem of Gromov from 1985:

  Theorem. *There are symplectic manifolds (M, ω) diffeomorphic to $\mathbb{R}^{2n}$ but not symplectomorphic to $(\mathbb{R}^{2n}, \omega_{std})$.*

  Elements of the proof. The main ingredient is showing that a compact Lagrangian submanifold of $\mathbb{C}^n$ must admit a non-constant holomorphic disc with boundary on it. This is shown solving the holomorphic curve equation explicitly for some complex structure and using continuation methods (elliptic theory) and compactness to produce the disc. Such a disc would pair positively with the Liouville form restricted to the Lagrangian, hence it could not be exact: so $\mathbb{C}^n$ has no closed exact Lagrangians. I am not aware if a proof exactly like I will sketch exists, but it is what intuitively happens: one can construct an immersed exact Lagrangian in $\mathbb{C}^n$ (these behave flexibly) and make it have only double points. Then, one resolves these: one finds a symplectic immersion $(\mathbb{C}^n, \omega) \to (\mathbb{C}^n, \omega_{\mathrm{std}})$ that lifts the closed exact immersed Lagrangian to an closed exact embedded Lagrangian. But this $(\mathbb{C}^n, \omega)$ contains a closed exact Lagrangian so it cannot be $(\mathbb{C}^n, \omega_{\mathrm{std}})$. $\square$

- The first examples of exotic Stein structures in $\mathbb{C}^{2n}$ (note the even complex dimension), $n \geq 2$, were constructed in 2005 by Seidel and Smith[3] by showing that certain Lagrangian tori were non-displaceable. In 2009 McLean used symplectic homology to distinguis bitween infinitely many pair-wise non-equivalent but diffeomorphic Stein structures on $\mathbb{C}^n$ and T^*M for $n \geq 4$. This was pushed to $n = 3$ by Abouzaid-Seidel. In $\mathbb{C}^2$ there can be no analogue since any finite type Stein manifold diffeomorphic to $\mathbb{R}^4$ is symplectomorphic to $\mathbb{R}^4$.[4] On the other hand, in 1998 Gompf constructs uncountably many non-finite type Stein manifolds which are homeomorphic to $\mathbb{C}^2$. We will focus on understanding some of the ingredients in McLean's proof.

1.3 The main tool

- Subcritical Stein manifolds have played some role in the seminar already, as they are as flexible as we would wish. Another phenomenon exhibiting flexibility is Hamiltonian displaceability (being able to apply a Hamiltonian isotopy to make two sets disjoint).

- Exact Lagrangians have also been mentioned. These, however, exhibit rigidity. In contrast to the last item, non-displaceability is an example of rigidity.

- Let $\mathrm{Exct}_{\mathrm{emb}}$ be the category of exact symplectic manifolds where the morphisms are exact symplectic embeddings. We want an invariant (indepednent of **exact** symplecto class and Liouville homotopy) that is able to somewhat encode the flexibility vs rigidity dichotomy. This takes the form of a functor

  $$F : \mathrm{Exct}_{\mathrm{emb}} \longrightarrow \mathrm{Alg},$$

  where Alg is some category of algebraic objects (e.g. $\mathbb{Z}/2$ or $\mathbb{Z}$-graded R-modules –even a graded commutative R-algebra– for some unital ring R). Here are some desirable properties concretizing the motto that "the algebra should only detect rigidity":

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[3] They in fact show a result that was known in algebraic geometry symplectically.

[4] This follows from the fact that the standard contact structure on $\mathbb{S}^3$ is the only tight one as well as the description of its Stein fillings through holomorphic discs.

a. If W is a Liouville subdomain of a Liouville manifold V, then $F(V) = 0$ implies $F(W) = 0$ (McLean).

Note that functoriality says $F(W) \to F(V) = 0$ which implies $F(W) = 0$ if the target category is of algebras (as $1 = 0$ iff the algebra is trivial).

b. Similarly, if we can hamiltonianly displace W from itself in V, then $F(W) = 0$.

c. Attaching subcritical handles to V should not change the value of $F(V)$ (Cieliebak).

Attaching critical ones will change the value in a way regulated by a subtle surgery exact sequence.

d. If V has an exact Lagrangian, then it must be that $F(V)$ is non-trivial.

- We will later construct such an invariant using holomorphic curves, which philosophically are the primordial witnesses to rigidity in symplectic geometry. It will be called symplectic homology and denoted $F = \mathrm{SH}_\bullet$. The main idea is to tailor Hamiltonian Floer theory to completions of Liouville domains. We can make the functor valued in algebras by the pair of pants product. Here are some facts and computations, one not too difficult, one quite difficult.

  1. There is a natural map $H_{\bullet+n}(V, \partial V) \longrightarrow \mathrm{SH}_\bullet(V)$, as in usual Hamiltonian Floer theory. Unlike it, though, it is not always an isomorphism. When it isn't, the Weinstein conjecture holds (Viterbo).
  2. The symplectic homology of the ball vanishes, $\mathrm{SH}_\bullet(B^n) = 0$. We can understand this as follows (though it is simple-ish computation): we can Liouville embedd a small ball into a very big one, and then displace the smaller one, so the claim would fallow from *b*. Consequently, $\mathrm{SH}_\bullet(V) = 0$ if V is subcritical. Putting this together with 1. we can deduce that the Weinstein conjecture holds for subcritically fillable contact manifolds.
  3. (Viterbo) For a smooth manifold M and $\mathbb{D}^*M$ its tangent disc bundle with the Liouville form, then $\mathrm{SH}_\bullet(\mathbb{D}^*M) \cong H_\bullet(\mathcal{L}M; R)$, where $\mathcal{L}M$ is the free Loop space of M. Intuitively one should think about this as follows: the Reeb flow on $\mathbb{S}^*M$ agrees with the geodesic flow (we have used a Riemannian metric to define disc bundle), so Reeb orbits and closed geodesics are in correspondance and the Morse theory of the energy functional can be related to $\mathrm{SH}_\bullet$ and classically constructs $\mathcal{L}M$. The isomorphism is, in fact, as R-algebras (intertwining the pair of pants and Chass-Sullivan products).

- We can already draw some nice (almost formal) consequences from this computations. First, recall that *a.* followed from the algebra structure and functoriality. Now, *d.* follows because if there is a closed exact Lagrangian L in V we have an exact embedding of a (sufficiently small) disc cotangent of L into V by the Weinstein neighborhood theorem. Hence, $\mathrm{SH}_\bullet(\mathbb{D}^*L) \to \mathrm{SH}_\bullet(V)$. Viterbo's theorem, however, implies the LHS does not vanish, so then the algebra $\mathrm{SH}_\bullet(V)$ cannot vanish. This also implies that exact Lagrangians cannot be displaced away from themselves! This also proves Gromov's theorem that subcritical Stein domains have no exact Lagrangians.

Given these properties, we will discuss some of the ideas of the theorem of McLean. After that we will roughly explain what symplectic homology is (without proofs).

2 The easy parts of McLean's theorem

We explain the construction in dimension 8 as it generalized in a relatively straighforward way. Consider the hypersurface $V := \{x^7 + y^2 + z^2 + w^2 = 0\} \subset \mathbb{C}^4$ and its complement $Z = \mathbb{C}^4 \setminus V$; we will study the variety X, a blowup of Z at infinity. More concretely, we blow up $\mathbb{C}^4$ at a smooth point $p \in V$, consider $\tilde{V}$ the proper transform of V and $X = \mathrm{Bl}_p(\mathbb{C}^4) \setminus \tilde{V}$. We can perform the end-connected sum of Stein domains by attaching a 1-handle along their boundaries, denote X_n the

n-fold self end-connected sum of X. Note that the complement of a codimension 2 thing enables a non-trivial fundamental group, that's why we blow up.

- First of all, X is diffeomorphic to $\mathbb{R}^8$. This comes from the fact that V is a Brieskorn sphere and its link is homeomorphic to $\mathbb{S}^5$ (classical result) and so it itself is homeomorphic to $\mathbb{R}^6$. Algebraic topology shows that X is also contractible (no fundamental group+no higher homology+Hurewicz and Whitehead). Using the first theorem[5] in the last section we deduce that X is diffeomorphic to $\mathbb{R}^8$, from which the fact that all the X_n are too follows.
- Let $\iota(\cdot)$ count the number of idempotents of $\mathrm{SH}_\bullet(\cdot)$, which is an invariant of the domain and potentially easier to compute than the full homology. Essentially, giving a new functor

  $$\mathrm{Exct}_{\mathrm{emb}} \xrightarrow{\iota} \{1, 2, 3, \ldots, \infty\}.$$

  If one can show that $\iota(X_n) \neq \iota(X_m)$ for $n \neq m$, one will have that their symplectic homologies are different and hence not equivalent.
- By *c.* above, we have $\mathrm{SH}_\bullet(X_n) = \mathrm{SH}_\bullet(X)^{\oplus n}$. In particular, $\iota(X_n) = \iota(X)^n$ and so it suffices to show that $1 < \iota(X) < \infty$.
- Recall that $\mathrm{SH}_\bullet$ is non-trivial iff $1 \neq 0$, and so iff $\iota \geq 2$, as both 1 and 0 are idempotents. Showing non-triviality is work but doable, showing finiteness of ι is the real hassle.
- It turns out that one can show that $\mathrm{SH}_\bullet(X)$ is ring-isomorphic to $\mathrm{SH}_\bullet(Z)$. Intuitively this comes from the idea of X being a blow up at infinity of Z, which can be roughly interpreted as a surgery operation. In reality, one needs to show that taking a Lefschetz subfibration (with some additional properties) does not change $\mathrm{SH}_\bullet$. This is hard work and involves defining a $\mathrm{SH}_\bullet^{\mathrm{lef}}$ suited to Lefschets fibrations (tailoring Hamiltonian Floer homology at infinity in a vertical and horizontal direction), understanding how it does not change when taking certain subfibrations and showing that it is isomorphic to the usual $\mathrm{SH}_\bullet$.
- Now, everything has been boiled down to computing the symplectic homology for a very specific divisor complement in $\mathbb{C}^4$. This is still hard work but doable.

  Symplectic homology can be graded by the group $H_1(Z)$ (orbits) and algebraic considerations imply that idempotents must be in $\mathrm{SH}_4(Z)$ and be torsion in $H_1(Z)$ (partly because of this, it is good to go from X, with no H_1, to Z). By carefully examining the example and definitions one notices that the Hamiltonian computing $\mathrm{SH}_\bullet$ can be taken to have finitely many orbits of index 4, so there can only be finitely many idempotents.

3 A bad sketch of Symplectic Homology

We explain how to construct $\mathrm{SH}_\bullet$, which we have putatively explored in the previous section. This is a bad sketch because I am going to avoid (almost) all technicalities just to give a fairy tale explanation.

- Floer introduces Hamiltonian Floer theory to solve the Arnold conjecture, which lower bounds the number of fixed points of a Hamiltonian symplectomorphism by the number of fixed points of a function on the manifold. Floer proves (in the symplectically aspherical case, for compactness) that the conjecture holds in the non-degenerate case. Given a (time-dependent) Hamiltonian H_t (whose time one flow is a given Hamiltonian symplecto), we consider its 1-periodic orbits (in bijective correspondence with fixed points of the symplecto) and assemble

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[5] It is not obvious that it should be of finite type. To show that the topology does not accumulate at infinity is somewhat of an algebraic fact: one can compactify adding an ample divisor and use a section of the corresponding positive line bundle that defines the divisor to get a Morse function on the complement. It will not have critical points near the divisor.

them into the vector space freely generated by them. One can then count rigid holomorphic cylinders between the orbits to get an endomorphism of this vector space and use the compactness theory of this cylinders to show that it squares to zero. The corresponding homology is called Floer homology. PDE-theoretic continuation methods then allow to show independence of the Hamiltonian chosen. Choosing then one that is $\mathcal{C}^2$-small the complex above is nothing more than the Morse complex, so its Floer homology is Morse homology. This implies the claim of the theorem.

- This construction does not come out of the blue, it is the culmination of a long effort to generalize Morse theory for the energy functional on the loop space of a Riemannian manifold. Say that (W, ω) is an exact symplectic manifold and λ a primitive of ω. The critical points of the functional

$$\begin{aligned} \mathcal{A}_H : \mathcal{C}^\infty(\mathbb{S}^1, W) &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto -\int_{\mathbb{S}^1} \gamma^*\lambda + H_t(\gamma(t))dt \end{aligned}$$

  are exactly the 1-periodic orbits. In fact, the L^2-gradient (after choosing a compatible almost-complex structure) of it $\nabla\mathcal{A}_H(\gamma) = J(\dot\gamma - X_H(\gamma))$. What made Morse theory for this functional possible was the generalization of a Morse index for it by Conley and Zehnder and Floer's insight of interpreting the gradient flow not as an ODE, but an elliptic PDE (Floer's holomorphic curve equation).

- To make Floer theory work on non-compact manifold one needs to make sure solutions do not escape to infinity. Here is where the convexity conditions come in. To create a somewhat canonical Floer homology for a Liouville domain that sees its qualitative symplectic geometry we must find a way to make a somewhat canonical choice of Hamiltonians that is tailored to completions and captures all Reeb orbits of ∂V.
  - We begin with an exercise. Take a contact manifold (M, α) and consider its symplectization $(\mathbb{R}_r \times M, \omega)$. The idea is to consider a Hamiltonians in the symplectization whose 1-periodic orbits are the Reeb orbits of the contact manifold of any period. Take a Hamiltonian H of the form $h(e^r)$ for a smooth function h with $h' \geq 0$. The Hamiltonian vector field for this is $h'(e^r)R$ where R is the Reeb vector field of α. So if we make sure that $h'(e^r)$ is always increasing and comes from 0 and goes to ∞ (from $r = -\infty$ to $r = \infty$) then the Reeb orbits of any period are in bijective correspondence to the 1-periodic orbits of X, the larger the period, the larger the height r.
  - The above is a local model for ∂V in the completed domain $\widehat{V}$. We now consider a Hamiltonian H^∞ that after sufficiently large r_0 in the cylindrical end looks like $h(e^r)$ with the extra properties on h just stated (as well as $\partial_r h(e^r)|_{r=r_0}$ being smaller than the smallest period of a closed Reeb orbit of α)[6] and it's negative in the interior of V. Then, $\mathrm{SH}_\bullet(V)$ can be defined as the Floer homology for $(\widehat{V}, H^\infty)$. This definition is a bit hard to work with analytically and, in practice, one ends up passing through the following one.
  - Alternative definitions usually consist of defining a certain class of admisisble Hamiltonians and take a direct of their Floer homologies. This is analytically quite convinient in terms of continuation maps as we can take families of Hamiltonians with more freedom than the definition above. Moreover, it works well when one is interested in quantitative properties. We call a Hamiltonian H τ-admissible if
    1. $H_t < 0$ in the interior of V,
    2. $H_t(r, y) = \tau e^r + c$ in $[r_0, \infty) \times \partial V$ for r_0 sufficiently large and c some constant,

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[6] This is to somehow separate the inherent symplectic topology of V (and Hamiltonians on it) from contact dynamics of M.

3. and non-degenerate.

If τ is not a period of a Reeb orbit (these are a countable set) then H_t cannot have 1-periodic orbits after r_0 and so it has finitely many. Given τ and τ'-admissible Hamiltonians H_t^τ and $H_t^{\tau'}$ for $\tau \leq \tau'$ we can consider a homotopy between them of a similar form so that slopes at infinity get steeper under the homotopy and obtain well-defined continuation maps

$$FH_\bullet(H_t^\tau), J \longrightarrow FH_\bullet(H_t^{\tau'}, J')$$

and consequently define their direct limit under an increasing sequence of admissible τ's:

$$\mathrm{SH}_\bullet(V) := \lim_{\tau \to \infty} HF_\bullet(H_t^\tau, J).$$

- If we choose H^τ to be a time-independent Morse function with no critical points outside of the interior or V, that is $\mathcal{C}^2$-small except in $[r_0, \infty) \times \partial V$ where we require it equals $H^\tau(r, y) = \tau e^r + c$, then its Floer homology is Morse homology, and since the gradient flow points outwards on ∂V, it computes the singular homology $H_{\bullet+n}(V, \partial V)$. In light of the continuation maps we get a map

  $$H_{\bullet+n}(V, \partial V) \longrightarrow \mathrm{SH}_\bullet(V).$$

  Note that if ∂V has no closed Reeb orbits then all continuation maps must be isomorphisms as the only information they contain is the one in the interior of V. This would imply that the map above is an isomorphism. We have sketched consequences of this above.