Introductory talk on Stein and Weinstein manifolds

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Abstract

The following is a writeup of the introductory talk for the student seminar *Symplectic geometry of Stein and Weinstein manifolds* at the HU Berlin in Winter Semester 2025–26. Its contents overlap heavily with the synopsis of the seminar available on my homepage, but it is not identical. It also contains a few things that I did not have time to say in the talk.

1 Some motivation

Weinstein manifolds are an important class of *open* symplectic manifolds that can be constructed in a topological fashion by assembling various standard pieces called "symplectic handles". Within symplectic geometry, they are important and arise naturally for several reasons:

- Many important results of differential topology based on Morse theory can be adapted into the symplectic world as results about Weinstein manifolds.
- One of the great open problems of symplectic topology is to identify purely topological conditions that will characterize whether or not a given closed manifold admits a symplectic form, or whether two given symplectic forms are deformation equivalent. For Weinstein manifolds, which are open symplectic manifolds with a bit of extra structure, these problems have known solutions that are elegant and deep. Conjectured strategies to attack the general problem on closed manifolds often make use of the known solution for Weinstein manifolds.
- By a deep result of Donaldson, every closed symplectic manifold X contains distinguished symplectic submanifolds $Y \subset X$ of codimension 2 such that $X \setminus Y$ is naturally a Weinstein manifold.
- Every compact Weinstein manifold, also known as a Weinstein *domain*, has a nonempty boundary that inherits a natural *contact* structure: Weinstein domains thus define a natural class of *symplectic fillings* of closed contact manifolds.
- Symplectic handles define a notion of *contact surgery* on contact manifolds along isotropic submanifolds, such that the contact manifolds before and after surgery are related by a so-called *Weinstein cobordism*. In light of this, one can use symplectic invariants of Weinstein manifolds (such as *symplectic homology* and other Floer-type invariants) to compute related invariants of contact manifolds.
- By a result of Giroux, every closed contact manifold admits an open book decomposition whose binding is a contact submanifold of codimension two, and whose pages carry natural Weinstein structures, making them symplectic fillings of the binding.

The following additional reason for interest in Weinstein manifolds is not purely symplectic, but instead lives at the interface between symplectic and *complex* geometry:

- By a deep theorem originally sketched by Eliashberg (and built on top of a complex-analytical result of Grauert), every Weinstein manifold can be endowed with a complex manifold structure such that it embeds properly and holomorphically into $\mathbb{C}^N$ for some $N \gg 0$. Complex manifolds with this property are called *Stein manifolds*.

2 Main definitions

Recall that a **symplectic** manifold (W,ω) is an even-dimensional manifold W carrying a symplectic form $\omega \in \Omega^2(W)$, meaning a 2-form that is closed ($d\omega = 0$) and nondegenerate ($\omega(X,\cdot) \neq 0$ for all vectors $X \neq 0$). A **Weinstein structure** on W is a tuple (ω, φ, X) consisting of three pieces of data:

- A symplectic form ω;
- A Morse function $\varphi : W \to \mathbb{R}$, i.e. a smooth function with isolated critical points at which the Hessian is nonsingular;
- A vector field X that satisfies

$$\mathcal{L}_X \omega = \omega \qquad \text{and} \qquad d\varphi(X) > 0 \text{ outside of critical points.}$$

The first condition on X makes it a **Liouville vector field**, so that its flow φ_X^t satisfies $(\varphi_X^t)^*\omega = e^t\,\omega$ for all t and is thus called a *symplectic dilation*. The second makes it a **gradient-like** vector field with respect to φ.[1] There is a unique 1-form λ defined by

$$\omega(X,\cdot) = \lambda,$$

called the **Liouville form** dual to X, and by Cartan's magic formula, the fact that X is a Liouville vector field is then equivalent to λ satisfying $d\lambda = \omega$. The symplectic form on a Weinstein manifold is therefore exact, so by Stokes' theorem, W cannot be closed; it is necessarily an open manifold, meaning that each of its connected components is either noncompact or has nonempty boundary. If $\partial W \neq \emptyset$, then we require one further condition, namely

$$d\varphi|_{T(\partial W)} \equiv 0,$$

so that φ is constant on each connected component of ∂W, and the gradient-like Liouville vector field X is therefore transverse to the boundary. One can show that the latter is equivalent to the condition that the 1-form λ restricts to ∂W as a *contact form*, and thus induces a *contact structure* on the boundary (and also on every other regular level set of φ).

We take this opportunity for a quick digression on contact geometry, which is often called the odd-dimensional cousin of symplectic geometry. On a manifold M of dimension $2n-1$, a **contact form** is a 1-form satisfying

$$\alpha \wedge (d\alpha)^{n-1} := \alpha \wedge \underbrace{d\alpha \wedge \ldots \wedge d\alpha}_{n-1} \neq 0,$$

i.e. the top-dimensional form $\alpha \wedge (d\alpha)^{n-1}$ is a volume form. The hyperplane distribution

$$\xi := \ker \alpha \subset TM$$

is in this case called a **contact structure** induced by α. The Frobenius integrability theorem implies that ξ is integrable if and only if $\alpha \wedge d\alpha \equiv 0$, thus the contact condition $\alpha \wedge (d\alpha)^{n-1} \neq 0$ can be thought of as a "maximal nonintegrability" condition. An equivalent way of writing it is that $d\alpha|_\xi$ defines a nondegenerate 2-form on the distribution ξ, and the contact condition is in this sense an odd-dimensional analogue of the nondegeneracy condition for a symplectic form ω on a $2n$-manifold W, which can also be written as $\omega^n \neq 0$.

As discussed above, the Liouville form $\lambda = \omega(X,\cdot)$ on a Weinstein manifold restricts to each regular level set of $\varphi : W \to \mathbb{R}$ as a contact form; in particular, it defines a contact form on ∂W. The following now provides a good reason to take interest in contact *structures* rather than contact forms: by a basic result in contact geometry known as **Gray's stability theorem**, any smooth 1-parameter family of contact structures $\{\xi_t\}_{t\in[0,1]}$ on a closed manifold M comes from a smooth isotopy, i.e. there exists a smooth family of diffeomorphisms $\{\varphi_t : M \to M\}_{t\in[0,1]}$ such that $\varphi_0 = \mathrm{Id}$ and $\varphi_t^*\xi = \xi$. For a specific situation in which this hypothesis arises, notice that for any given symplectic form ω and Morse function $\varphi : W \to \mathbb{R}$, gradient-like Liouville vector fields are not unique, but the set

$$\left\{X \in \mathcal{X}(W) \;\middle|\; \mathcal{L}_X\omega = \omega \text{ and } d\varphi(X) > 0\right\}$$

is convex. It follows that any two choices of such vector fields can be related by a smooth family of them, giving rise to a smooth family of contact forms on each level set $\varphi^{-1}(*)$. By Gray's theorem, the

[1] Strictly speaking, the most useful definition of the term "gradient-like" also requires X and φ to satisfy a technical condition at the critical points of φ, which is explained in [CE12], though this detail is often ignored in papers on the subject.

induced contact structures are all isotopic, and therefore diffeomorphic. To summarize: Up to isotopy, the contact structure induced on each regular level set of the Morse function on a Weinstein manifold (and in particular on the boundary) depends only on the level set and the symplectic form nearby—it does not depend on the choice of gradient-like Liouville vector field. In this sense, **Weinstein domains** (meaning compact manifolds with boundary endowed with Weinstein structures) form a natural class of **symplectic fillings** of closed contact manifolds.

3 Handles and Morse theory

The original motivation for Weinstein structures comes from a symplectic variant of classical Morse theory, in which real-valued functions are used to produce handle decompositions, and in this way to understand the topology of smooth manifolds. We now briefly discuss how this works in the more general setting of smooth topology, without symplectic structures.

If W is a compact smooth n-manifold with boundary $M := \partial W$ containing an embedded $(k-1)$-sphere

$$S^{k-1} \cong L \subset M$$

for some $k \in \{1, \dots, n\}$, one can produce a new manifold W' by attaching an n-dimensional **k-handle** along a neighborhood $N(L) \subset M$ of L,

$$W' := W \cup_{N(L)} (\mathbb{D}^k \times \mathbb{D}^{n-k}),$$

where the attachment identifies $S^{k-1} \times \mathbb{D}^{n-k} \cong N(L) \subset M$ with $\partial \mathbb{D}^k \times \mathbb{D}^{n-k} \subset \partial(\mathbb{D}^k \times \mathbb{D}^{n-k})$. The boundary $M' := \partial W'$ of the new manifold constructed in this way can be described as the result of a surgery operation on M, in which $N(L) \cong S^{k-1} \times \mathbb{D}^{n-k}$ is replaced by a copy of $\mathbb{D}^k \times \partial \mathbb{D}^{n-k} = \mathbb{D}^k \times S^{n-k-1}$ using the obvious identification

$$\partial(S^{k-1} \times \mathbb{D}^{n-k}) = S^{k-1} \times S^{n-k-1} = \partial(\mathbb{D}^k \times S^{n-k-1}).$$

The k-dimensional disk $\mathbb{D}^k \times \{0\} \subset \mathbb{D}^k \times \mathbb{D}^{n-k}$ is called the **core** of the handle, whose boundary gets identified with the so-called **attaching sphere** $S^{k-1} \cong L \subset M$. Dually, the new manifold M' contains a distinguished $(n-k-1)$-sphere called the **belt sphere**, which is the boundary of the **cocore** $\{0\} \times \mathbb{D}^{n-k} \subset \mathbb{D}^k \times \mathbb{D}^{n-k}$ of the handle. Note that this construction also makes sense for $k = 0$ if W is taken to be the empty set: attaching an n-dimensional 0-handle thus means replacing the empty set with a disk $\mathbb{D}^n$. Dual to this, if $M = \partial W \cong S^{n-1}$, then one can attach an n-handle along the attaching sphere $L := M$, which means capping off ∂W with a disk to produce a closed manifold.

In classical Morse theory as described in [Mil63], each Morse function $\varphi : W \to \mathbb{R}$ on a smooth n-manifold W gives rise to a handle decomposition of W, in which there is a one-to-one correspondence between k-handles for $k \in \{0, \dots, n\}$ and critical points of φ with **Morse index** k, meaning points at which φ can be written in the form

$$\varphi(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2 + \text{const}$$

in suitable coordinates. For reasonable choices of gradient-like vector fields X, the **stable manifold**

$$W_p^- := \left\{ x \in W \;\middle|\; \lim_{t \to \infty} \text{flow}_X^t(x) = p \right\}$$

of a critical point $p \in \text{Crit}(\varphi)$ can be regarded as the core of the corresonding handle, and dually, its **unstable manifold**

$$W_p^+ := \left\{ x \in W \;\middle|\; \lim_{t \to -\infty} \text{flow}_X^t(x) = p \right\}$$

is the cocore of the handle.

4 Symplectic handles

The symplectic version of this story begins with an observation originally due to Weinstein [Wei91]: for each $k = 0, \dots, n$, one can define $2n$-dimensional **symplectic k-handles**

$$(\mathbb{D}^k \times \mathbb{D}^{2n-k}, \omega, \varphi, X),$$

equipped with not only a symplectic form ω but also a Morse function φ having a unique critical point of index k at $(0,0) \in \mathbb{D}^k \times \mathbb{D}^{2n-k}$, whose gradient $X := \nabla\varphi$ is a Liouville vector field pointing transversely inward at $\partial\mathbb{D}^k \times \mathbb{D}^{2n-k}$ and outward at $\mathbb{D}^k \times \partial\mathbb{D}^{2n-k}$. The construction is easy to describe. Let us identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$ and write its coordinates as

$$z = (z^- = x^- + iy^-, z^+ = x^+ + iy^+) \in \mathbb{C}^k \times \mathbb{C}^{n-k} = \mathbb{C}^n = \mathbb{R}^{2n}.$$

By reordering the coordinates, this has an obvious identification with

$$\mathbb{R}^k \times \mathbb{R}^{2n-k} \ni \big(y^-, (x^-, x^+, y^+)\big),$$

in which our handle can be defined as the subset $\mathbb{D}^k \times \mathbb{D}^{2n-k}$. The desired Morse function with a critical point of index k can now be defined as

$$\varphi(z) := -\frac{1}{2}|y^-|^2 + |x^-|^2 + \frac{1}{4}|z^+|^2.$$

The chosen factors in front of each term are justified by the following computation. Composing the differential of φ with the standard complex structure of $\mathbb{C}^n$ gives the 1-form

$$\lambda := -d\varphi \circ i = \sum_j \left(y_j^- \, dx_j^- + 2x_j^- \, dy_j^-\right) + \frac{1}{2}\sum_j \left(x_j^+ \, dy_j^+ - y_j^+ \, dx_j^+\right),$$

which is a primitive of the standard symplectic form

$$\omega := d\lambda = \sum_j \left(dx_j^- \wedge dy_j^-\right) + \sum_j \left(dx_j^+ \wedge dy_j^+\right).$$

One easily checks that the Liouville vector field X satisfying $\omega(X,\cdot) = \lambda$ is simply the gradient of φ, so in particular, it is a gradient-like vector field, and symplectic handles therefore come naturally with Weinstein structures. We notice that the core of the handle coincides with the stable manifold of the critical point, which is an **isotropic** submanifold, meaning that ω vanishes on its tangent spaces. Dually, the cocore coincides with the unstable manifold of the critical point, and is a **coisotropic** submanifold, meaning that each of its tangent spaces contains its symplectic orthogonal complement.

The symplectic handles described above can be used as basic building blocks to construct Weinstein manifolds by handle attachment: the precise construction requires the use of standard neighborhood theorems that describe a collar neighborhood of any boundary component of a symplectic manifold with a transverse Liouville vector field, so that the Weinstein structure of a given manifold W can be glued smoothly to the Weinstein structure of the handle $\mathbb{D}^k \times \mathbb{D}^{2n-k}$. Conversely, any Weinstein manifold (W, ω, φ, X) can be decomposed into symplectic handles that are in bijective correspondence with the critical points of the Morse function φ, with cores taking the form of stable manifolds of critical points and cocores taking the form of unstable manifolds.

5 Existence of Weinstein structures

You may have missed the following detail in the above discussion: a $2n$-dimensional symplectic k-handle can be defined for each $k = 0, \ldots, n$, but *not* for $k = n+1, \ldots, 2n$. There are good geometric reasons for this restriction: as mentioned above, the core of a Weinstein handle is an isotropic submanifold, as is the stable manifold of any critical point in a Weinstein manifold, while dually, cocores of Weinstein handles and unstable manifolds of critical points in Weinstein manifolds are coisotropic submanifolds. By basic symplectic linear algebra, isotropic submanifolds in a symplectic $2n$-manifold can have dimension at most n; this is why symplectic k-handles are only possible for $k \le n$. It means there is a significant topological restriction on the $2n$-dimensional manifolds that admit Weinstein structures: such a manifold must admit a Morse function whose critical points all have index at most n, making it retractible to an n-dimensional CW-complex, a property that $2n$-manifolds do not typically have. It is a remarkable fact—and one of the main results that we will study in this seminar—that for $n > 2$, the existence of such a Morse function is not just necessary, but very nearly also *sufficient*:

Theorem. *For $n > 2$, a $2n$-dimensional manifold W admits a Weinstein structure if and only if it admits a nondegenerate 2-form (or equivalently an almost complex structure) and an exhausting Morse function whose critical points all have index at most n.*

This theorem is a *flexibility* result, also known as an *h-principle*, because it says that the obvious minimal topological conditions necessary for the existence of a certain geometric structure (in this case a Weinstein structure) are also sufficient.

6 Complex geometry and Stein manifolds

The picture becomes more remarkable with a bit of complex geometry added into the mix. Classically, a **Stein manifold** is a (necessarily noncompact) complex manifold (W, J) that admits a proper holomorphic embedding into $\mathbb{C}^N$ for some $N \gg 0$. Here, we denote by $J : TW \to TW$ the associated almost complex structure, which defines multiplication by i on the complex tangent spaces, but we stress that J is assumed to be an *integrable* almost complex structure, meaning it comes from an atlas of complex coordinate charts whose transition maps are holomorphic. For any properly and holomorphically embedded Stein manifold $W \subset \mathbb{C}^N$, restricting the function $\varphi : \mathbb{C}^n \to \mathbb{R} : z \mapsto |z|^2$ to W produces a smooth function $\varphi : W \to \mathbb{R}$ with the following two properties:

- It is **exhausting**, i.e. it is proper and bounded below;
- It is **plurisubharmonic**, also called **J-convex**, which means that the 2-form

$$\omega_\varphi := -d(d\varphi \circ J) \in \Omega^2(W)$$

  satisfies $\omega_\varphi(X, JX) > 0$ for all vectors $X \neq 0$.[2] In particular, ω_φ is a symplectic form that *tames* the complex structure of W.

By a complex-analytical result of Grauert (which sadly goes beyond the scope of this seminar), the existence of such a function characterizes Stein manifolds: a complex manifold (W, J) is a Stein manifold *if and only if* it admits an exhausting J-convex function $\varphi : W \to \mathbb{R}$. Since J-convexity is an open condition, φ can also be assumed to be a Morse function without loss of generality. It is not difficult to show that in this situation, the gradient $X := \nabla\varphi$ with respect to the Riemannian metric $g := \omega_\varphi(\cdot, J\cdot)$ is a Liouville vector field on the symplectic manifold (W, ω_φ), thus making $(\omega_\varphi, \varphi, \nabla\varphi)$ a Weinstein structure on W. Here is the result that inspires the title of [CE12]:

Theorem. *Every Weinstein structure is homotopic to one that arises in the manner described above from a Stein structure. Moreover, two Stein structures are homotopic as Weinstein structures if and only if they are homotopic as Stein structures.*

Corollary. *For $n > 2$, a $2n$-manifold can be made into a Stein manifold if and only if it admits both an almost complex structure and an exhausting Morse function whose critical points all have index at most n.*

Note that in general, it is considered a quite difficult problem to determine whether a given smooth (real) manifold of even dimension can be made into a complex manifold—the case of S^6 for example is famously open.[3] The corollary above shows however that this problem becomes tractable, and has a quite simple solution in the language of algebraic topology, as soon as one adds in a bit of extra data beyond the complex structure.

7 Flexibility

A final important result worth mentioning is that for $n > 2$, there is also a special class of Weinstein structures called **flexible**, whose classification up to homotopy satisfies an h-principle. The flexible Weinstein structures notably include those which are **subcritical**, meaning that the Morse function φ has only critical points with indices *strictly* less than n. Emmy Murphy's discovery [Mur] of **loose Legendrian submanifolds** in 2012 revealed that there are also flexible Weinstein structures containing critical handles, and in fact, every Weinstein structure is homotopic via nondegenerate 2-forms to a *unique* one that is flexible:

Theorem. *Two flexible Weinstein structures on a manifold W of dimension $2n > 4$ are homotopic as Weinstein structures if and only if their underlying nondegenerate 2-forms are homotopic as nondegenerate 2-forms (or equivalently, almost complex structures).*

[2] Plurisubharmonicity can also be described as the condition that φ restricts to a strictly subharmonic function on every complex curve.

[3] An obvious necessary condition for a real manifold to be made into a complex manifold is that it must admit an *almost* complex structure, a condition that can be characterized completely in algebro-topological terms using characteristic classes and obstruction theory. For S^6, it is easy to write down an explicit almost complex structure, but easy examples one can write down are not integrable. The question of which almost complex manifolds can actually be made into complex manifolds is, in general, wide open.

For all three of the theorems stated above, the general idea behind the proof is to work with handle decompositions, developing symplectic analogues of constructions that originate in classical Morse theory. This includes some constructions that first appeared in the proof of Smale's h-cobordism theorem, on which the solution to the higher-dimensional Poincaré conjecture was based.

References

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