①

Real analytic manifolds

Goal: "holomorphically attach handles"

↳ we need to approximate smooth objects by real analytic ones (why, how?)

Definition: (REAL ANALYTIC)

$f: U \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^m$ is real analytic if it is locally given by a convergent power series.

i.e. $\sum_{n=0}^{\infty} a_n (x-p)^n$ near a point $p \in U$

⇒ real analytic manifolds: atlas s.t. all transition f. are real analytic

r.a. submanifold: is locally the transverse zero set of a real analytic function

↳ i.e. the level set $\{f(x)=0\}$ intersects transversely with direction of increase ⇝ $\nabla f \neq 0$

⇒ ensuring a $(n-1)$-submanifold

⇒ real analytic bundles and sections are defined analogously

[→ Rmk: ODE with real analytic coeff. depends only real analytically on all parameters
[5.39]]

②

Complexification

REAL ANA $\xrightarrow{\#^{\mathbb{C}}}$ HOLOMORPHIC natural functor called complexification

note: $f: U \subset \mathbb{R}^n \to \mathbb{C}^m$ can be uniquely extended to

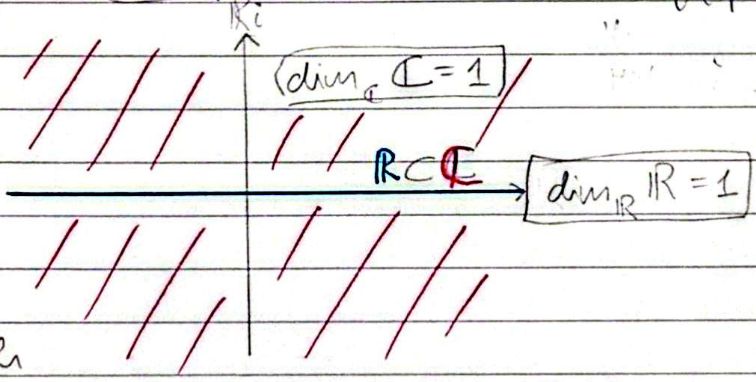
$f^{\mathbb{C}}: U^{\mathbb{C}} \subset \mathbb{C}^n \to \mathbb{C}^m$ with $U^{\mathbb{C}} \cap \mathbb{R}^n = U$. More generally:

lemma: V, W be complex mfds and $M \subset V$ a real analytic totally real real submfd with $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} V$.

[5.40]

complex structure $J(T_pM) \cap T_pM = 0$

Then the real anal. map $f: M \to W$ extends uniquely to a holomorphic map



$f^{\mathbb{C}}: \mathrm{Op}M \to W$ with $\mathrm{Op}M$ suff small in V.

If additionally $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W$ and f is a r.a. diffeo. of M onto totally real $N \subset W$, then $f^{\mathbb{C}}: \mathrm{Op}M \underset{\text{hol}}{\overset{\text{hol}}{\rightleftarrows}} \mathrm{Op}N$ is a biholomorphism.

proof: $p \in M$. pick real analytic coord chart $\phi: \mathbb{R}^n \supset U_1 \to M$ and a holomorphic coord chart $\psi: \mathbb{C}^n \supset U_2 \to V$ both mapping $0 \mapsto p$.

complexify: $(\psi^{-1} \circ \phi)^{\mathbb{C}} = \tilde{\Phi}: \mathbb{C}^n \supset U_1^{\mathbb{C}} \to \mathbb{C}^n$ biholomorphism

Then $\Phi = \psi \circ \tilde{\Phi}: \mathbb{C}^n \supset U \to V$ is holomorphic coord chart mapping $U \cap \mathbb{R}^n$ to M.

Now pick a holomorphic coord chart $\Psi: \mathbb{C}^m \supset U' \to W$ near $f(p)$

Complexify: $(\Psi^{-1} \circ f \circ \Phi)^{\mathbb{C}} = \tilde{F}: U \to \mathbb{C}^m$

then $F = \Psi \circ \tilde{F} \circ \Phi^{-1}$ is a holomorphic extension of f to a nbhd of p in V.

By uniqueness of hol. extensions, this does not depend on the chosen charts on V and W. + the extensions around different points of M fit together to the $f^{\mathbb{C}}$ thats desired.

the last statement follows from imp. func. thm + $(\text{real iso})^{\mathbb{C}} = \text{complex iso}$

$\square$

Thm (Fundamental) theorem (Bruhat and Whitney) [5.41]

Any real analytic manifold M has a complexification, i.e., a complex mfd. $M^{\mathbb{C}}$ with $\dim_{\mathbb{C}} M^{\mathbb{C}} = \dim_{\mathbb{R}} M$ which contains M as a totally real submanifold.
The germ of a complexification $M^{\mathbb{C}}$ is unique in the following sense: V, W complex mfds cont M as a real analytic and totally real submfd with $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W = \dim_{\mathbb{R}} M$, then $\exists$ nbhds of M in V and W which are biholomorphic.

Sketch: Construction of $M^{\mathbb{C}}$:

1. pick a locally finite covering of M by countably many real analytic ~~totally real submfds~~ coordinate charts

$$\phi_i : \mathbb{R}^n \supset U_i \to M \quad \text{s.t.}$$

the transition functions

$$\phi_{ij} := \phi_j^{-1} \circ \phi_i : U_{ij} := \phi_i^{-1}(\phi_i(U_i) \cap \phi_j(U_j)) \to U_{ji}$$

are real analytic diffeos.

2. Construct open subsets $U_i^{\mathbb{C}} \subset \mathbb{C}^n$ with $U_i^{\mathbb{C}} \cap \mathbb{R}^n = U_i$ and $U_{ij}^{\mathbb{C}} \subset U_i^{\mathbb{C}}$ sim. s.t. the ϕ_{ij} extends to biholomorphic maps $\phi_{ij}^{\mathbb{C}} : U_{ij}^{\mathbb{C}} \to U_{ji}^{\mathbb{C}}$ satisfying the following cond:

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<u>cocycle conditions:</u>

(i) $\phi_{ij}^{\mathbb{C}} = (\phi_{ji}^{\mathbb{C}})^{-1}$ and $\phi_{ii}^{\mathbb{C}} = \mathrm{Id}|_{U_{ii}^{\mathbb{C}} = U_i^{\mathbb{C}}}$

(ii) $\phi_{ij}^{\mathbb{C}}$ maps $U_{ijk} = U_{ij}^{\mathbb{C}} \cap U_{ik}^{\mathbb{C}}$ biholomorphically onto $U_{jik}^{\mathbb{C}}$ and $\phi_{jk}^{\mathbb{C}} \circ \phi_{ij}^{\mathbb{C}} = \phi_{ik}^{\mathbb{C}} : U_{ijk}^{\mathbb{C}} \to U_{kij}^{\mathbb{C}}$

3. Define $M^{\mathbb{C}}$ as $\coprod_i U_i^{\mathbb{C}} / \sim$ with $z_i \sim z_j \Leftrightarrow$ $z_i \in U_{ij}^{\mathbb{C}}$ and ~~...~~ $z_j = \phi_{ij}^{\mathbb{C}}(z_i) \in U_{ji}^{\mathbb{C}}$

(2. cocycle conditions make this into an equivalence relation)

4. The incl. $U_i^{\mathbb{C}} \hookrightarrow \coprod_j U_j^{\mathbb{C}}$ induce coord charts $U_i^{\mathbb{C}} \hookrightarrow M^{\mathbb{C}}$ with biholomorphic transition fn.

(we can choose $U_i^{\mathbb{C}}, U_{ij}^{\mathbb{C}}$ carefully enough to ensure Hausdorffness)

5. Uniqueness follows from the lemma. □

note: As a real mfd. a (suff. small) $M^{\mathbb{C}} \underset{\text{diffeo}}{\cong_{\mathbb{R}}} TM$.
(since M is totally real we have

$$TM \xrightarrow{\cdot i} \nu(M) \text{ is an isomorphism}$$
$$\cong (iT_pM \, \forall p)$$

(since our M is half dimensional + totally real)

5

Collection of results

Complexification has functorial properties.

↳ e.g. $N \subset M$ real analytic submfd

$\Rightarrow$ $N^{\mathbb{C}} \subset M^{\mathbb{C}}$ (as long as $N^{\mathbb{C}}$ is suff. small)

Observation (Grauert) [$M^{\mathbb{C}}$ are Stein for real analy. mfds]

[5.42] Proposition: Let $M^{\mathbb{C}}$ be a complexification of a real analytic manifold M. Then M posesses arbitrarily small nbhds in $M^{\mathbb{C}}$ which are Stein.

example: $\mathbb{R}P^n$ has a "maximal" complexification

$$(\mathbb{R}P^n)^{\mathbb{C}} = \mathbb{C}P^n \quad \text{which is compact so}$$

not Stein, but smaller tubular regions in $\mathbb{C}P^n$ are Stein.

proof: 1. M posesses arbitrary small nbhds with exhausting ~~[illegible]~~ J-convex functions. (Prop. 2.15)

2. Grauert theorem: J-convex $\Rightarrow$ Stein in classical sense (5.17)

We call a complexification $M^{\mathbb{C}}$ which is Stein a GRAUERT TUBE. □

Corollary [5.43]: Every real analytic manifold admits a proper embedding (real analytic) into some $\mathbb{R}^N$.

proof: Grauert tubes $M^{\mathbb{C}}$ admit proper holomorphic embedding into $\mathbb{C}^{2n+1}$ (5.15), restrict this embedding to M. □

Corollary [5.44]: $P \subset N \subset M$ properly embedded real analytic submanifolds and let $d \in \mathbb{Z}_{\geq 0}$. Then for every real analytic function $f: N \cup Op(P) \to \mathbb{R}$ $\exists$ a real analytic $F: M \to \mathbb{R}$ with $F|_N = f$ whose d-jet coincides with that of f in P.

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proof: Let $M^{\mathbb{C}}$ be a Grauert tube of M. (After shrinking) we may assume
$$P^{\mathbb{C}} \subset N^{\mathbb{C}} \subset M^{\mathbb{C}}$$ and f complexifies to a holomorphic function $f^{\mathbb{C}} : N^{\mathbb{C}} \cup Op(P^{\mathbb{C}}) \to \mathbb{C}$.
Use Corollary 5.37 (coherent sheaves) $\rightsquigarrow$ provide a hol. fun. $G : M^{\mathbb{C}} \to \mathbb{C}$ with $G^{\mathbb{C}}|_{N^{\mathbb{C}}} = f^{\mathbb{C}}$ and whose d-jet agrees with that of $f^{\mathbb{C}}$ at points of $P^{\mathbb{C}}$.
Restrict to the real part. $\square$

Corollary (5.45): Every properly embedded real analytic submanifold N of a real analytic manifold M is the common zero set of a finite number [max. $2(\dim_{\mathbb{R}} M+1)(\operatorname{codim}_{\mathbb{R}} N+1)$] of real analytic functions $f_i : M \to \mathbb{R}$ such that for all $x \in N$, the differentials $d_x f_i : T_x M \to \mathbb{C}$ satisfy
$$\bigcap_i \ker d_x f_i = T_x N.$$

proof: Complexify: $N^{\mathbb{C}} \subset M^{\mathbb{C}}$ where we again chose a Grauert tube $M^{\mathbb{C}}$.
Corollary 5.38 says, $N^{\mathbb{C}}$ is the zero set of at most $(\dim_{\mathbb{R}} M+1)(\operatorname{codim} N+1)$ of real analytic functions $F_i : M^{\mathbb{C}} \to \mathbb{C}$ satisfying the condition.
Restrict ~~to~~ $\operatorname{Re} F_i$ and $\operatorname{Im} F_i$ to M to get f_i. $\square$

5.36 Cartan Theorem: V Stein, $\mathcal{F}$ coherent analytic sheaf on V.
Then
(A) $\forall x \in V$, $H^0(V, \mathcal{F})$ generates $\mathcal{F}_x$ as an $\mathcal{O}_x$-module
(B) $H^q(V, \mathcal{F}) = \{0\}$ $\forall q > 0$

both Corollaries are literally what was used above but replace $M^{\mathbb{C}}$ with a Stein mfd. W.

ろ

to conclude this section there is an extension to our lemma from before (for the people covering §16)

Lemma (extension to the non-totally real case)
Let U, V, W be complex mfds and $M \subset V$ a real analytic totally real submfd. with $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} V$.
Then any real analytic map $f: U \times M \to W$ whose restriction to $U \times m$ is holomorphic $\forall m \in M$ extends uniquely to a holomorphic map $f^{\mathbb{C}}: U \times V \supset Op(U \times M) \to W$.

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§5.8 REAL ANALYTIC APPROXIMATION

[Analytic Extensions of diff. functions defined in closed sets]

We have: Corollary about proper real analytic embedding + [thm Whitney]

$\Rightarrow$ Proposition: (every C^k-function can be approximated by a C^k-real analytic one)

[.48] Let $f: M \to \mathbb{R}$ be a C^k-function on a real analytic manifold.
Then for every pos. continuous function $h: M \to \mathbb{R}_+$ $\exists$ a real analytic function $g: M \to \mathbb{R}$ s.t.

$$|D^k g(x) - D^k f(x)| < h(x) \quad \forall x \in M.$$

We equip M with a metric and connection so that we can write down (covariant) derivatives and norms $D^k f$ = (vector of derivatives up to k)

proof: Embed M real analytically into some $\mathbb{R}^N$ (Cor.).
Extend f to a C^k-function $F: \mathbb{R}^N \to \mathbb{R}$ and h to a cont. function $H: \mathbb{R}^N \to \mathbb{R}_+$.
Apply the aforementioned thm. by Whitney $\rightsquigarrow$ $\exists$ a real analytic function $G: \mathbb{R}^N \to \mathbb{R}$ s.t. $|D^k G(x) - D^k F(x)| < H(x)$ $\forall x \in \mathbb{R}^N$.
Restrict back.

$\square$

⑧

We can generalize this to sections in analytic fiber bundles.

Corollary: [5.49] Let $f: M \to E$ be a C^k-section of a real analytic fiber bundle $E \to M$ over r.a.m. M. Then for every positive cont. function $h: M \to \mathbb{R}_+$ $\exists$ a real analytic section $g: M \to E$ s.t.

$$|D^k g(x) - D^k f(x)| < h(x) \quad \forall x \in M.$$

↳ example: This implies the following: every Riemannian metric on M r.a.m. can be C^k-approximated by a real analytic metric.

Remembering the remark [5.39] about solutions to ODE with r.a. coefficients depending only r.a. on all parameters.
↳ exponential map is real analytic
↳ real analytic tubular and collar nbhds
↳ this allows us to extend any compact r.a.m. with bdry to a slightly larger open one.

Now: 5.49 and 5.44 give a general approximation result and an extension result.
We want to combine both.

Lemma [5.51]: Consider a real analytic vector bundle $E \to N$, integers $k \geq d \geq 0$, a continuous function $h: E \to \mathbb{R}_+$.
Let $f: E \to \mathbb{R}$ be a smooth function whose $\boxed{\text{fibrewise } d\text{-jet}}$ $\boxed{J^d f}$ is real analytic. Then $\exists$ a smooth function $g: E \to \mathbb{R}$ and arbitrary small open nbhds $U \subset V$ of N in E with

(i) $j^d g = j^d f$ along N
(ii) $|D^k g(x) - D^k f(x)| < h(x) \quad \forall x \in E$
(iii) g is real analytic on U and $g = f$ outside V

note: If E is real analytic so is $J^d E$.
$J^d f$: every C^{k+d}-function induces a C^k-section $J^d f$ of $J^d E$ which we call fibrewise d-jet.
by taking Taylor of degree d. (Also: "jet prolongation")

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<u>Proof of [5.51]</u>: Consider the real analytic bundle $J^{k\,id}E \to N$ whose fibers at $x \in N$ consist of all sums of monomials on E_x of degree $(d+1 \to k)$.

Let $J^d f = J^{k\,id} f = J^k f$ be the section of $J^{k\,id}E$ defined by f.
By [5.49] $\exists$ a real analytic section F of $J^{k\,id}E$ with

$$|D^k F(x) - D^k J^{k\,id} f(x)| < h(x) \quad \forall x \in N.$$

So $G := J^d f + F : E \to \mathbb{R}$ is a real analytic function with $J^d G = J^d f$ and $|D^k G(x) - D^k f(x)| < h(x) \quad \forall x \in N$.

This holds still on nbhd of N in E, we can interpolate $G \to f$ outside a smaller nbhd to obtain g.

$\square$

Next consider a properly embedded $N \subset M$ (r.a.m).
Any real analytic Riemannian metric on M. Its exponential map yields a real analytic diffeo $\Phi : Op(\underset{\text{of } E \to N}{0\text{-section}}) \to Op(N \subset M)$

We define the NORMAL d-jet along N of $f : M \to \mathbb{R}$ as the fiberwise d-jet of $f \circ \Phi$.

$\Rightarrow$ <u>Corollary [5.52]</u> (replacing
- vector bundle $\to$ fiber bundle $E \to M$
- real valued functions $\to$ sections
- $N \subset M$ properly embedded
- fiberwise d-jet $\to$ normal d-jet)

Same as above. [5.51]

$\Rightarrow$ <u>Theorem [5.53]</u>: Consider a real analytic fiber bundle $E \to M$, a properly embedded r.a.m. $N \subset M$, $k \geq d \geq 0$, $h : M \to \mathbb{R}_+$.
Let $f : M \to E$ be a smooth section whose normal d-jet $J^d f$ along N is real analytic. Then $\exists$ a real analytic section $F : M \to E$ with

(i) $j^d F = j^d f$ along N
(ii) $|D^k F(x) - D^k f(x)| < h(x) \quad \forall x \in M$.

(no small nbhds anymore!)

proof: Step 1: [5.52, 5.44]

- suffices to show the case of a real-valued function $f: M \to \mathbb{R}$.
- we may assume f is smooth (after C^k-approx. fixing normal d-jet)
- apply [5.52] and assume now that f is real analytic in a nbhd. of N.
- Pick any $l \geq k \geq d$. By [5.44] $\exists$ a r.a. function $H: M \to \mathbb{R}$ whose l-jet coincides with ~~f~~ that of f at points of N.
- $g := f - H$ vanishes to order ~~d~~ l along N ~~and sats~~ ~~$|D^k G - D^k g| < h \ \forall x \in M$~~
- ~~$F := G + H$~~ • Suppose we find a r.a. section $G: M \to \mathbb{R}$ that vanishes to order d along N and sat.

$$|D^k G - D^k g| < h \quad \forall x \in M.$$

$\Rightarrow$ $F := G + H : M \to \mathbb{R}$ is the desired approxim. ~~extension~~ of f

and

$$|D^k F - D^k f| = |D^k G + D^k H - D^k f| = |D^k G - D^k g| < h \ ✓$$

Step 2: [5.45], [5.48]

- suffices to show for $f: M \to \mathbb{R}$ vanishing on order $l = 2d + k + 1$ along N.
- cor [5.45] $\exists$ r.a. functions $\phi_1, \dots, \phi_m: M \to \mathbb{R}$ such that $N = \{\phi_1 = \dots = \phi_m = 0\}$ and $\bigcap \ker d_x \phi_i = T_x N \quad \forall x \in N$.
  Then $\phi := \phi_1^2 + \dots + \phi_m^2 : M \to \mathbb{R}$ is real analytic and $N = \phi^{-1}(0)$.
- $\mathrm{Hess}_x \phi$ is positive definite in directions transverse to N
  $\Rightarrow$ $\phi \gtrsim \mathrm{dist}_N^2$ for the distance from N wrt to Riem metr on M.

note: in a nbhd of $p \in N$: $|f(x)| \leq C_p \, \mathrm{dist}(M, x)^l$ and hence $|f(x)||\phi^{-1}(x)|^{-d} \leq C_p \, \mathrm{dist}(M, x)^{l-2d} = C_p \, \mathrm{dist}(M, x)^{k+1}$.

↳ $g := f \phi^{-d}$ defines a C^k-function on M.

- Prop. [5.48] $\exists$ r.a. function $G: M \to \mathbb{R}$ such that $|D^k G - D^k g| < h/(1 + \phi^d)$ on M.

$F := G \phi^d : M \to \mathbb{R}$ is the desired function.

□

§5.3 Definition of Stein manifolds:

Affine: A complex manifold V is Stein if it admits a proper holomorphic embedding into some $\mathbb{C}^N$.

J-convex: A complex manifold V is Stein if it admits an exhausting J-convex function (*) $f: V \to \mathbb{R}$.
↳ bounded below + proper
↳ preimages of cpt sets are cpt.

classic: A complex manifold V is Stein if it has the following properties:

(i) V is holomorphically convex (*)
(ii) $\forall x \in V$ $\exists$ holomorphic functions $f_1, \dots, f_n : V \to \mathbb{C}$ which form a holomorphic coordinate system at x
(iii) $\forall x \neq y \in V$ $\exists$ holomorphic function $f: V \to \mathbb{C}$ with $f(x) \neq f(y)$

note: (i) + (ii) $\Rightarrow$ (iii) and (i) + (iii) $\Rightarrow$ (ii)

overview:

$$\begin{tikzcd} & \text{affine} \arrow[dl] \arrow[dr, shift left] & \\ J\text{-convex} \arrow[rr, "5.17\ (\text{Grauert})"'] & & \text{classical} \arrow[ul, shift left, "5.15\ (\text{Bishop, Narasiham})"'] \end{tikzcd}$$

remark: Forster about the optimal N s.t. Stein mfds can be embedded. $N = [\frac{3n}{2}] + 1$ proven by Eliashberg, Gromov, Schürmann

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Definition: (J-convex): (V, J) almost complex structure J on smooth manifold $\rightsquigarrow$ J is integrable (i.e. $\Leftrightarrow$ $N(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y] = 0$.)
$\Rightarrow (V, J)$ complex mfd. and J is complex structure.
Now a function $\phi: V \to \mathbb{R}$ on an almost complex mfd. is called J-convex if

$$\omega_\phi (X, JX) > 0 \text{ for all nonzero tangent vectors } X.$$

where $\omega_\phi := -d(d\phi \circ J) \in \Omega^2(V)$
$= -dd^{\mathbb{C}}\phi = -dd\phi(J\cdot)$

note: If ω_ϕ is J-invariant, it defines a unique Hermitian form

$$H_\phi = \underbrace{\omega_\phi(\cdot, J\cdot)}_{=: g_\phi} - i\,\omega_\phi$$

↳ ϕ is J-convex $\Leftrightarrow$ H_ϕ is positive definite.

observe: In Euclidean space a function is convex if its Hessian is positive definite

now. example: $f : \mathbb{C} \to \mathbb{R}$, $f(z) = |z|^2 = x^2 + y^2$ is i-convex. hence restricted to a complex direction (real) convex.

· level sets are circles and restrictions $f(z)|_{\mathbb{C}}$ = parabolas

---

Holomorphic Convexity §5.1

Definition: $K \subset V$ subset of a complex manifold, define its HOLOMORPHIC HULL in V.

$$\hat{K}_V := \{x \in V \mid |f(x)| \leq \max_K |f| \text{ for all holomorphic functions } f : V \to \mathbb{C}\}$$

- depends on manifold V
- If $U \subset V$ open subset containing K $\Rightarrow$ $\hat{K}_U \subset \hat{K}_V$
- For $V = \mathbb{C}^n$ we can equivalently replace holomorphic functions by polynomials
  ↳ we call $\hat{K}_{\mathbb{C}^n}$ the POLYNOMIAL HULL
  If $\hat{K}_{\mathbb{C}^n} = K$ then we call K POLYNOMIALLY CONVEX

example: [5.1]
· Any $K \subset \mathbb{C}^n$ compact convex set has $\hat{K}_{\mathbb{C}^n} = K$.
since $\forall$ point $z \notin K$ $\exists$ complex linear function $\ell : \mathbb{C}^n \to \mathbb{C}$ s.t. $\mathrm{Re}(\ell(z)) > \max_{w \in K} \mathrm{Re}(\ell(w))$. Hence $|e^{\ell(z)}| > \max |e^{\ell(w)}|$.

Definition A complex mfd V is called HOLOMORPHICALLY CONVEX if $\hat{K}_V$ is compact for all compact $K \subset V$.

example [5.2]

- any open convex $\Omega \subset \mathbb{C}^n$ is hol. convex
  since Ω can be exhausted by compact convex set $K^i \subset \mathring{K}^{i+1}$
  $\rightsquigarrow$ $K^i \subset \hat{K}^i_\Omega \subset \hat{K}^i_{\mathbb{C}^n} = K^i$

- properly embedded complex submfds of V are holomorphically convex if V is

Definition: A compact $K \subset V$ is called HOLOMORPHICALLY CONVEX if it can be presented as an intersection of holom. convex open domains in V.

In other words K holomorphically convex if it has arbitrarily small open hol. conv. nbhds. (maybe non-std def)

note: If K satisfies $\hat{K}_V = K \Rightarrow$ hol. convex.
! converse $\Leftarrow$ is not true!
$\hookrightarrow$ 5.14 (upcoming)

Theorem [5.4] (Oka, Weil) A hol convex compact subset $K \subset \mathbb{C}^n$ is polynomially convex $\Leftrightarrow$ every holomorphic function on Op K can be approximated uniformly by polynomials on K.

$\hookrightarrow$ generalizes to [5.18]. A holomorphically convex compact subset K of a Stein mfd V satisfies $\hat{K}_V = K \Leftrightarrow$ every holomorphic function on Op K can be approx. uniformly on K by holom. functions on V.

$\hookrightarrow$ even further to sections of any holom vector bundle over a Stein mfd.

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§5.2 Relation to J-convexity

: J-convexity := strict J-convexity

Remark [5.5] Suppose V admits a (exhausting) J-convex function (e.g. V = open subset of $\mathbb{C}^n$ or of a Stein mfd.). Then any exhausting weakly J-convex fn $\phi: V \to \mathbb{R}$ can be turned into an exhausting smooth J-convex fn. ($\to$ §3)
On general smooth complex mfds this might not work

Theorem [5.6] For any open set $U \subset \mathbb{C}^n$ tfae:

(a) U is holomorphically convex
(b) $-\log \operatorname{dist}_{\partial U}$ is weakly i-convex in U
(c) U admits an exhausting i-convex function

note (b) $\Rightarrow$ (c) follows from remark [5.5]

$$\begin{tikzcd} & (a) & \\ (b) \arrow[rr, "{[5.5]}"'] & & (c) \arrow[ul, "\text{Levi problem}"'] \end{tikzcd}$$

↳ generalized in [5.7]

Theorem [5.7] Suppose the complex mfd. (V, J) admits an exh. J-convex function $\phi: V \to \mathbb{R}$.
Then all sublevel sets of ϕ satisfy

$$\widehat{\{\phi \leq c\}}_V = \{\phi \leq c\}$$

In particular (V, J) is holomorphically convex.

Now: Construct exh J-convex functions on bdd domains.
$\to$ Lemma [5.8], [5.9] (p. 91 incl. proofs)
$\Rightarrow$ Corollary [5.10]:

Any compact domain $W \subset \mathbb{C}^n$ with smooth weakly i-convex boundary admits an exhausting i-convex function on the interior

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[5.8] Let V be a complex mfd which possesses a J-convex function, and let $W \subset V$ be a compact domain with smooth J-convex boundary. Then $\exists$ a J-convex function $\phi : W \to \mathbb{R}$ which is constant on ∂W. In particular $\mathring{W}$ admits an exh. J-conv. function.

Define: PLURISUBHARMONIC HULL: K compact!

$$\hat{K}_V^{psh} := \{ x \in V \mid \phi(x) \leq \max_K \phi \text{ for all cont. weakly } J\text{-conv. } \phi : V \to \mathbb{R} \}$$

Proposition [5.12] Suppose V admits an exhausting J-convex function. Then $\hat{K}_V^{psh} = \hat{K}_V$ for every compact set $K \subset V$.

proof: $\hat{K}_V^{psh} \subset \hat{K}_V$ is clear bcs. $|f|^2$ is weakly J-conv for every holom. f.
The other direction, take $x \notin \hat{K}_V^{psh}$.

- $\exists$ weakly J-conv. $\phi : V \xrightarrow{\text{cont.}} \mathbb{R}$ with $\phi(x) > \max_K \phi$.
- after adding and smoothing, we may assume ϕ is exh., smooth, strict J-conv.
- pick regular value c of ϕ with $\max \phi < c < \phi(x)$
- Then [5.7]: $\hat{K}_V \subset \widehat{\{\phi \leq c\}}_V = \{\phi \leq c\}$ does not contain x.

$\square$

Proposition: [5.13] Suppose (V, J) admits an exhausting J-convex function. Then a compact subset $K \subset V$ satisfies $\hat{K}_V^{psh} = K$
$\Leftrightarrow$ $\exists$ exh. smooth weakly convex function $\psi : V \to \mathbb{R}_{\geq 0}$ such that $\psi^{-1}(0) = K$ and ψ is J-convex outside K.

proof: Construction $\to$ book (p. 92).

Corollary: For a closed totally real submfd. $L \subset \mathbb{C}^n$, there hold

(a) L is holomorphically convex

(b) If $\dim L = n$, it is not polynomially convex.

proof: (a) [2.15] $\Rightarrow$ dist_L^2 is i-convex on some nbhd $\not\subset$ U of L
so by Thm [5.7] $\hat{L}_U = L$ and L hol. conv. ✓

(b) symplectic geometry:

- Suppose L is polynomially convex.
- Let $\psi: \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ be the exh. function with $\psi^{-1}(0) = L$ provided by [5.13].

- After replacing ψ near L by smooth $\max(\psi, \varepsilon \text{dist}_L^2)$, $\varepsilon > 0$
$\rightarrow$ assume ψ is i-convex
- use [2.11] to make ψ completely exhausting

- [11.22] $\Rightarrow$ $-dd^{\mathbb{C}}\psi$ is diffeomorphic to the ω_{std} on $\mathbb{C}^n$
$= -d(d\psi \circ J)$
- ~~contradicts~~ Now L is exact Lagrangian for $-dd^{\mathbb{C}}\psi$ i.e. $-d^{\mathbb{C}}\phi|_L \equiv 0$.

  ↯ Gromovs thm. that $\nexists$ no closed exact Lagrangian submfds for ω_{std} on $\mathbb{C}^n$.

$\square$