Symplectic and Contact Preliminaries

Symplectic vector spaces

Defn A <u>symplectic vector space</u> (V, ω) is a real vector space V along w/ a non-degen. skew-symmetric bilinear form ω, ie $\omega(v, w) = -\omega(w, v)$ (skew-sym)

& $V \longrightarrow V^*$: $v \longmapsto \omega(v, -)$ (non-deg) is an isomorphism.

Obs: Skew-symmetry & non-degeneracy force V to be even-dimensional.

Let $\dim V = k$ and $e_1, \ldots, e_k$ be a basis of V. Define a matrix $A_{k \times k}$ by $A_{ij} = \omega(e_i, e_j)$

Skew-sym $\Rightarrow A^T = -A \Rightarrow \det(A) = (-1)^k \det A)$

Non-deg $\Rightarrow \det(A) \neq 0$ $\quad \therefore k$ must be even. **Notn:** $\dim V = 2n$

Defn: A linear map $\Psi : (V_1, \omega_1) \longrightarrow (V_2, \omega_2)$ between sympl vector spaces is called <u>symplectic</u> (or a <u>symplectomorphism</u>) if $\Psi^* \omega_2 = \omega_1$

ie $\omega_2(\Psi v, \Psi w) = \omega_1(v, w) \quad \forall v, w \in V_1$

Obs: A symplectomorphism must be an isomorphism.

Ex: $(\mathbb{R}^{2n}, \omega_{std})$ where $\omega_{std} := \sum_{i=1}^{n} dx_i \wedge dy_i$ where $x_1, x_2, \ldots, x_n, y_1, \ldots, y_n$ is the standard basis of $\mathbb{R}^{2n}$ & dx_i, dy_i's are their dual vectors.

Lemma: Every symplectic vector space $\overset{\dim V = 2n}{(V, \omega)}$ has a basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ st $\omega(x_i, y_i) = 1$, $\omega(x_i, x_j) = \omega(y_i, y_j) = 0 \;\; \forall i, j$ & $\omega(x_i, y_j) = 0$ for $i \neq j$.

ie. $\exists$ symplectomorphism $(V, \omega) \rightarrow (\mathbb{R}^{2n}, \omega_{std})$. Such a basis is called a <u>symplectic basis</u>.

Pf: Induction on n. ∎

Defn: Given $W \subset V$, a linear subspace of V ((V, ω) is a symp VS) the <u>ω-orthogonal complement</u> / <u>symplectic complement</u> is defined as

$$W^\omega := \{ v \in V \mid \omega(v, w) = 0 \;\; \forall w \in W \} \subseteq V$$

In contrast with inner products, where the orthogonal complements of subspaces satisfy $W \cap W^\perp = \{0\}$, we don't necessarily have $W \cap W^\omega = \{0\}$. W is called

→ <u>isotropic</u> if $W \subseteq W^\omega$

→ <u>co-isotropic</u> if $W^\omega \subseteq W$

→ <u>symplectic</u> if $W \cap W^\omega = \{0\}$, in which case ω restricts to a symplectic form on W making $(W, \omega|_W)$ a sympl VS.

→ <u>Lagrangian</u> if $W = W^\omega$

However we still have the following (as in inner product spaces)

Lemma: $\dim W + \dim W^{\omega} = \dim V$, $(W^{\omega})^{\omega} = W$

Pf: $\iota_\omega : V \to V^*$: $v \mapsto \omega(v, -)$ is an iso, & identifies W^{ω} with the annihilator of W in V^*, which has complementary dim.
$\Rightarrow \dim W + \dim W^{\omega} = \dim V$

$W \subseteq (W^{\omega})^{\omega}$ & $\dim W = \dim (W^{\omega})^{\omega} \Rightarrow (W^{\omega})^{\omega} = W$

Cor: Let $W \subseteq (V, \omega)$ & $\dim V = 2n$. W: isotropic $\Rightarrow \dim W \leq n$, W coisotropic $\Rightarrow \dim W \geq n$

W Lagrangian $\Rightarrow \dim W = n$. W: Lagrangian $\Leftrightarrow$ W: isotropic and $\dim W = n$

W: isotropic $\Leftrightarrow$ W^{ω} coisotropic. W: symplectic $\Leftrightarrow$ W^{ω}: symplectic. ∎

Lemma: ① Any isotropic subspace is contained in a Lagrangian subspace.
② Any basis of a Lagrangian subspace can be extended to a symplectic basis.

Pf: $W \subset V$ isotropic $\Rightarrow$ $W \cup \{v\}$ is isotropic for any $v \in W^{\omega} \setminus W \Rightarrow$ ①
For ②, argue in $\mathbb{R}^{2n}$. $W \subseteq \mathbb{R}^{2n}$ Lagrangian $\Rightarrow W' := J_0 W$ Lagrangian where $J_0 = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$

$\omega(x,y) = \langle J_0 x, y \rangle_{\mathbb{R}^{2n}}$ so W' can be identified w/ W^* via ι_ω.
$\therefore$ We can pick a "dual" basis in W'. ∎

From this, we can get normal forms for the different kinds of subspaces.
If $W \subseteq V$ is isotropic, then $\exists$ a symplectic basis $x_1, \dots, x_n, y_1, \dots, y_n$ of V st
$W = \text{span}\{x_1, \dots, x_k\}$ for some $k \leq n$.
Similarly for other types.
If $W \subseteq V$ is symplectic, a symplectic basis of W can be extended to one of V.

Ex: → Every 1-dimensional subspace of (V, ω) is isotropic, every hyperplane is co-isotropic.
→ $\text{span}\{x_1, \dots, x_n\}$ & $\text{span}\{y_1, \dots, y_n\} \subseteq (\mathbb{R}^{2n}, \omega_{std})$ are Lagrangian.

Defn: A complex structure on a real vector space V is a linear map $J : V \to V$ st $J^2 = -Id_V$
→ This gives V a complex vector space structure by $(a+ib)v := av + bJv$.
It follows that only even-dim'l vector spaces can have cpx structures

Defn: A complex structure J on (V, ω) is said to be compatible (/ ω-compatible) if
$\omega(Jv, Jw) = \omega(v, w)$ $\forall v, w \in V$, and $\omega(v, Jv) > 0$ $\forall v \in V \setminus \{0\}$

Rmk: This condition is equivalent to requiring that $g_J := \omega(\cdot, J\cdot)$ is an inner product on V. Note that the standard cpx str on $\mathbb{R}^{2n}$ in $\mathbb{C}^n$ is compatible w/ ω_{std}

Defn: In the above defn, J is said to be tamed by ω if $\omega(v, Jv) > 0$ $\forall v \in V \setminus \{0\}$
For any such J, $g_J(v,w) := \frac{1}{2}[\omega(v, Jw) + \omega(w, Jv)]$ is an inner product on V.

Any cpx str J on (V,ω) compatible w/ ω also gives a Hermitian metric on (V,J) via $H(v,w) := \omega(v,Jw) - i\,\omega(v,w)$ (*)

Def (+) We call such a pair (ω, J) a Hermitian str & (V,ω,J) a Hermitian vector space. (w/ H defined as above) (given V, J, H, ω is the imaginary part of H in (*))

Recall from last time: A real subspace $W \subset V$ of a cpx vs (V,J) is called
→ totally real if $W \cap JW = \{0\}$ → totally coreal if $W + JW = V$
→ maximally real if $W \cap JW = \{0\}$ and $W + JW = V$
→ complex if $W = JW$

For a real subspace of a Hermitian vs (V,ω,J), let $W^\perp := \{v \in V \mid g_J(v,w) = 0 \ \forall w \in W\}$
Then we can relate the previous types of subspaces of (V,ω) by

Lemma: (a) W isotropic $\Leftrightarrow JW \subset W^\perp \Rightarrow W$ totally real
(b) W coisotropic $\Leftrightarrow W^\perp \subset JW \Rightarrow W$ totally coreal
(c) W lagrangian $\Leftrightarrow JW = W^\perp \Rightarrow W$ maximally real
(d) W complex $\Rightarrow W$ symplectic

Pf: Observe that $W^\omega = (JW)^\perp = J(W^\perp)$

W is isotropic $\Leftrightarrow W \subseteq W^\omega = (JW)^\perp$
$\Leftrightarrow JW \subseteq W^\perp$ and then $W \cap JW \subseteq W \cap W^\perp = \{0\}$

Similarly, other statements also follow.

Symplectic vector bundles

Defn: A symplectic vector bundle (E,ω) is a vector bundle $E \to B$, whose fibers (E_x) are equipped with symplectic forms (ω_x) varying smoothly w/ $x \in B$, ie ω is a smooth section of $\Lambda^2 E^* \to B$ st each $\omega_x \in \Lambda^2 E_x^*$ is a symplectic form on E_x. ω is called a sympl str on E

Define $\mathcal{J}(E,\omega) := \{ J \in \mathcal{J}(E) \mid g_J(v,w) := \omega(v,Jw)$ is a Euclidean bundle metric$\}$
→ complex structures on E (on each fiber, smoothly varying)
"compatible complex str" / "Hermitian str"
and in $\Gamma(\mathrm{End}(E))$
equip it w/ C^∞_{loc}-topology.

Conversely, given a complex vector bundle (E,J), define
$\Omega(E,J) := \{\omega$ sympl str on $E \mid g_J(v,w) := \omega(v,Jw)$ is a Euclidean bundle metric$\}$

also equipped w/ C^∞_{loc} topology.

Lemma: $\Omega(E,J)$ is non-empty & contractible.

Pf: A complex vector bundle has a Hermitian metric $H: E \oplus E \to \mathbb{C}$ & the imaginary part of H defines a symplectic structure on E. $\therefore \Omega(E,J)$ is non-empty.
Contractibility follows because it is a convex subset of $\Gamma(\Lambda^2 E^*)$.

Prop: $\mathcal{J}(E,\omega)$ is non-empty & contractible.

Pf: Let $\mathcal{M}et(E)$ denote the space of (smooth) Euclidean bundle metrics on E w/ C^∞_{loc} topology.

We had a cts map $\mathcal{J}(E,\omega) \xrightarrow{\Psi} \mathcal{M}et(E)$. We define $\mathcal{M}et(E) \xrightarrow{\Phi} \mathcal{J}(E,\omega)$

$J \longmapsto g_J$ st $\Phi \circ \Psi = \mathbb{1}_{\mathcal{J}(E,\omega)}$

which will give both non-emptiness (due to that of $\mathcal{M}et(E)$ & Φ) & contractibility of $\mathcal{J}(E,\omega)$ (again due to that of $\mathcal{M}et(E)$)

Let $g \in \mathcal{M}et(E)$

Non-degeneraty of g & ω imply that $\exists!$ invertible $A \in \Gamma(End(E))$ st $\omega \equiv g(A\cdot, \cdot)$.

Skew-symmetry of $\omega \Rightarrow A^T = -A \Rightarrow A$ commutes w/ $AA^T = A^TA$

A^TA is a positive operator, hence possesses a unique positive square root P,

& P commutes with an operator B if P commutes with A^TA

Define $J_g = (AA^T)^{-1/2}A = P^{-1}A$

$$J_g^2 = P^{-1}AP^{-1}A = P^{-2}A^2 = (A^TA)^{-1}A^2 = (-A^2)^{-1}A^2 = -\mathbb{1}$$

$\therefore$ We can define $\Phi(g) = J_g$. (compatibility can be checked using the definitions easily) Continuity of Φ follows from the continuity of the square root

Also, $\Phi(g_J) = J$.

$\therefore \Phi \circ \Psi = \mathbb{1}_{\mathcal{J}(E,\omega)}$ & this completes the proof.

Cor: The Chern classes $c_k(E,\omega) := c_k(E,J) \in H^{2k}(B)$ for a $J \in \mathcal{J}(E,\omega)$ do not depend on J.

Symplectic manifolds

Defn: A symplectic manifold (M,ω) is a smooth manifold along with a 2-form $\omega \in \Omega^2(M)$ that is closed and non-degenerate.
$(d\omega = 0)$

$f: (M,\omega) \to (M',\omega')$ a smooth diffeo is called a symplectomorphism if $f^*\omega' = \omega$

Ex: $(\mathbb{R}^{2n}, \omega_{std})$

Rmk: ω_p is a sympl form on T_pM when (M,ω) is a sympl mfd.
$(\forall p \in M)$

Each (T_pM, ω_p) is isomorphic to $(\mathbb{R}^{2n}, \omega_{std})$ from a lemma before.

In fact, Symplectic Darboux's thm states that (M,ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_{std})$

Rmk Since $d\omega_{std} \equiv 0$, this couldn't have been true if $d\omega \neq 0$.

Ex: Cotangent bundle of any smooth manifold M has a canonical symplectic str.

Let $\pi: T^*M \to M$ be the cotangent bundle projection.
$(\alpha \in T_q^*M) \mapsto q$

$\rightsquigarrow T\pi: T(T^*M) \to TM$ which maps $T_\alpha(T^*M)$ linearly to T_qM where $\alpha \in T_q^*M$

Define $\lambda \in \Omega^1(T^*M)$ by $\lambda_\alpha(\xi) := \alpha(T\pi(\xi))$ for each $\xi \in T_\alpha(T^*M)$

Then $\omega := d\lambda$ is a symplectic form on T^*M.

$\hookrightarrow$ If $\mathcal{U} \xrightarrow{x} \mathbb{R}^n$ is a chart for some $\mathcal{U} \subseteq M$, we can get a local expression for λ as follows.
$x = (x^1, \dots, x^n)$

We choose coordinates $(q^1, \dots, q^n, p^1, \dots, p^n)$ on $T^*M|_{\mathcal{U}}$ using this.

Define $q^i(\alpha) = x(q)$ where $\alpha \in T_q^*M$ ie $q^i = x \circ \pi$

& $p^i(dx^j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ & extend linearly to all 1-forms over $\mathcal{U}$.

$\frac{\partial}{\partial p^j} \in V(T^*M)$ $\forall j = 1, \dots, n$ & $\frac{\partial}{\partial q^j}$ satisfy $\pi_*\left(\frac{\partial}{\partial q^j}\right) = \frac{\partial}{\partial x^j}$ for $j = 1, \dots, n$

$\Rightarrow \lambda_\alpha\left(\frac{\partial}{\partial p^j}\right) = 0 \ \forall \alpha \in T^*M|_{\mathcal{U}}$

$\Rightarrow \lambda_\alpha\left(\frac{\partial}{\partial q^j}\right) = \alpha\left(\frac{\partial}{\partial x^j}\right) = p^j$

$\therefore \lambda = \sum_{j=1}^n p^j dq^j$ in these coords where $\alpha = (q^1, \dots, q^n, p^1, \dots, p^n)$ in local coords

$\Rightarrow \omega := d\lambda = \sum_{j=1}^n dp^j \wedge dq^j$ is a symplectic form on T^*M.

Defn: An almost complex structure J on a manifold M is a complex structure on its tangent bundle TM. $J \in \Gamma(\mathrm{End}(TM))$ st $J^2 = -\mathbb{1}$

If M is a complex manifold, then there is a natural almost cpx str on it, but acs can exist even when the mfd has no cpx str.
If J comes from an atlas of complex charts, then J is said to be integrable.

→ (ω, J) on M is compatible if $\omega(\cdot, J\cdot)$ is a Riemannian metric on M.
$\Omega^2(M)$ acs on M

If $N \subset M$ is an almost cpx submfd, then N is also a symplectic submfd of M with $(\omega|_N, J|_N)$ a compatible pair.

→ A Kähler mfd is a symplectic manifold (M, ω) with an integrable almost complex str J compatible w/ ω.

∴ Kähler mfds are all complex mfds.

Ex: $(\mathbb{C}^n, \omega_{std})$ where $\mathbb{C}^n$ is identified w/ $\mathbb{R}^{2n}$ via $z_j = x_j + iy_j$

$(\mathbb{CP}^n, \omega_{FS})$ where ω_{FS} is obtained from ω_{std} on $\mathbb{C}^{n+1}$ by restricting to S^{2n+1}

$(\mathbb{C}^{n+1} \setminus \{0\}) / (\mathbb{C} \setminus \{0\}) = S^{2n+1}/S^1$

The 2-form descends to a well-defined one on the quotient S^{2n+1}/S^1.
ω_{FS} is compatible w/ std cpx str on $\mathbb{CP}^n$.

Complex submanifolds of Kähler manifolds are also Kähler.
∴ Stein manifolds are Kähler.
Smooth projective varieties are Kähler.

Suppose (V, J) is an almost cpx mfd & $\phi : V \to \mathbb{R}$ is a J-convex function,
(i.e $\omega_\phi := -d(d\phi \circ J) \in \Omega^2(V)$ satisfies $\omega_\phi(X, JX) > 0 \ \forall X \neq 0$)
then ω_ϕ is symplectic. If J is integrable, then ω_ϕ is compatible w/ J.

⇒ J-convex functions on Stein manifolds induce sympl forms compatible w/ J.

Nijenhuis tensor: An almost cpx str J on a mfd V gives a tensor $N \in \Gamma(T^1_2 M)$

$$N(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

Newlander & Nirenberg proved that an almost cpx str J is integrable if & only if the corresponding $N \in \Gamma(T^1_2 M)$ vanishes identically.

Moser's deformation trick and applications

Suppose $\{\omega_t\}_{t\in[0,1]}$ is a family of symplectic forms on a manifold M and we want to know whether there exist a smooth family of diffeomorphisms $\phi_t : M \to M$ (diffeotopy) such that $\phi_t^* \omega_t = \omega_0 \ \forall t \in [0,1]$.

The idea here is to look for a time-dependent vector field $\{X_t \in \mathfrak{X}(M)\}_{t\in[0,1]}$ whose time-t flows are ϕ_t. We derive sufficient conditions on X_t such that the required condition holds and verify that those conditions can be satisfied.

Differentiating $\phi_t^* \omega_t = \omega_0$ wrt t gives

$$\underbrace{\phi_t^* \mathcal{L}_{X_t} \omega_t}_{} + (\phi_t)^* \dot{\omega}_t = 0 \quad \text{where } \dot{\omega}_t = \frac{\partial}{\partial t}\omega_t$$

|| Cartan's magic formula

$$d(\iota_{X_t}\omega_t) + \underbrace{\iota_{X_t} d\omega_t}_{=0 \text{ as } \omega_t \text{ is symplectic.}}$$

$\therefore$ It is sufficient to have X_t st $d(\iota_{X_t}\omega_t) = -\dot{\omega}_t$ $\quad (**)$

Thm: (Moser's stability theorem) Let M be a mfd st $\partial M = \phi$, and let $\{\omega_t\}_{t\in[0,1]}$ be a family of symplectic forms on M st $\omega_t|_{K^c} \equiv \omega_0|_{K^c} \ \forall t \in [0,1]$ for some $K \subseteq M$ compact,

compactly supp whose class

$$\& \ [\omega_t - \omega_0] = 0 \in H^2_{c\,dR}(M) \ \forall t \in [0,1]$$

Then $\exists$ a diffeotopy ϕ_t wt $\phi_t = Id$ outside a cpct set st $\phi_t^* \omega_t = \omega_0$.

Pf: $\omega_t - \omega_0 = d\beta_t$ for some $\beta_t \in \Omega^1_c(M)$ compactly supp

β_t can be chosen to vary smoothly w/ t & st all are supported in some compact $K \subseteq M$, by an argument of Banyaga. See [Ban].

Then $\dot{\omega}_t = d\dot{\beta}_t$ and from $(**)$,

it suffices to find X_t st $\iota_{X_t}\omega_t = -\dot{\beta}_t$

$$\| $$

$$\omega_t(X_t, -) = -\dot{\beta}_t \in \Omega^1(M)$$

Non-degeneracy of $\omega_t \Rightarrow X_t$ exist.

Since $\dot{\beta}_t \equiv 0$ outside K, $X_t \equiv 0$ outside K, and hence $\phi_t = Id$ outside K and ϕ_t exists for all t. ∎

Cor: If M is a closed mfd w/ a smooth family of sympl forms $\{\omega_t\}_{t\in[0,1]}$ st $\forall t\in[0,1]$, $[\omega_t]=[\omega_0]\in H^2_{dR}(M)$, then $\exists\ \phi_t$ diffeotopy st $\phi_t^*\omega_t=\omega_0$ st $\phi_0=Id$.

can be ensured because ϕ_t can be obtained as a flow

Darboux's theorem can be obtained as a special case of symplectic neighbourhood theorem but we prove it separately here since there is a simpler proof.

Thm (Darboux's thm) Let (M,ω) be a symplectic manifold and $p\in M$ be any point. Then $\exists$ a nbhd of p, $\mathcal{U}\subseteq M$ and a symplectomorphism $\varphi:(\mathcal{U},\omega|_{\mathcal{U}})\to(\mathbb{R}^{2n},\omega_{std})$

[i.e. "All symplectic manifolds locally look the same". Recall that Riemannian mfds do not have this property - (M,g) cannot be locally isometric to $(\mathbb{R}^n,g_{std})$ where the Riemann tensor does not vanish.]

Pf: By choosing a chart around $p\in M$, it suffices to prove that if ω is a sympl form on $\mathbb{R}^{2n}$ (as a mfd), then $\exists$ nbhds $U,U'\subset\mathbb{R}^{2n}$ of 0 and a symplectomorphism $\varphi:(U,\omega|_U)\to(U',\omega_{std}|_{U'})$.

We may assume WLOG that $\omega(0)=\omega_{std}$, by composing w/ a symplecto.

Define $\omega_t=(1-t)\omega_{std}+t\omega$. $d\omega_t=0\ \forall t\in[0,1]$, and since $\omega_t(0)=\omega_{std}\ \forall t\in[0,1]$ and non-degeneracy is an open condition, $\exists$ a nbhd U_1 of 0 st $\omega_t|_U$ is a sympl form on $U_1\ \forall t\in[0,1]$. We may also take U_1 to be contractible, so that ω_t are all also exact. Let $\omega=d\lambda$, and define

$$\lambda_{std}:=\sum_{i=1}^n y_i\,dx_i$$

$$\lambda_t:=(1-t)\lambda_{std}+t\lambda \Rightarrow \omega_t=d\lambda_t$$

$$\dot\omega_t=\omega-\omega_{std}=d(\lambda-\lambda_{std})=d\dot\lambda_t$$

From (**), it suffices to find $\mathcal{X}(\mathcal{U})\ni X_t$ st $\omega_t(X_t,\cdot)=-\dot\lambda_t$, which exist due to non-degeneracy of ω_t.

Note that we are not yet done, since the flow might not exist for all $t\in[0,1]$. However, we know $\dot\lambda_t(0)=0$, which implies $X_t(0)=0\ \forall t\in[0,1]$. So the flow through 0 exists $\forall t\in[0,1]$, and hence the flow through points in some neighbourhood U exist for all $t\in[0,1]$.

We can choose this U & take $\varphi=\phi_1^X$. ∎

Prop: (Extension of Moser's Stability theorem) Let W be a compact mfd w/ boundary $\partial W\neq\emptyset$. If $\{\omega_t\}_{t\in[0,1]}$ is a smooth family of sympl forms st ω_t all agree on $T(\partial W)$ and the relative cohomology classes $[\omega_t-\omega_0]\in H^2_{dR}(W,\partial W)$ are all zero. Then $\exists\phi_t$ diffeotopy st $\phi_t|_{\partial W}=Id$ & $\phi_t^*\omega_t=\omega_0$.

defined via $\Omega^*(W,\partial W)$ consisting of forms α on W that vanish on ∂W ie $i^*\alpha=0$ where $\partial W\overset{i}{\hookrightarrow}W$

Pf: Choose a collar nbhd $\partial W\times[0,1)$ around ∂W, htpy equiv to ∂W. The operator P used in the pf of htpy invariance of H^*_{dR} gives a smooth family $\alpha_t=P(\dot\omega_t)$ of 1-forms on $[0,1)\times\partial W$ vanishing along ∂W st $\boxed{d\alpha_t=\dot\omega_t}$

$\partial W\times[0,1)\ \underset{\pi}{\overset{i_0}{\rightleftarrows}}\ \partial W\times[0,1)$

$\alpha_t(\partial W)=0$; if $\dot\omega_t$ ie.

Choose a cutoff function f on W which has cpct supp in $\partial W\times[0,1)$, & is $\equiv1$ near ∂W. Then $\dot\omega_t-d(f\alpha_t)$ have cpct support in Int W and $[\dot\omega_t-d(f\alpha_t)]\in H^2_{c,dR}(\text{Int}\,W)$ are zero. We apply Moser's Stability thm to these to get ϕ_t. $\phi_t|_{\partial W}=Id_{\partial W}$ as the vector fields will vanish on ∂W.

$[\dot\omega(t)=d\beta_t+d(f\alpha_t)$ for some $\beta_t]$

∎

The following lemma will be useful to prove a proposition on symplectic normal forms which leads to neighbourhood theorems that follow:

Lemma: Let W be a compact submanifold of a manifold V. If ω_0, ω_1 are symplectic forms on V that agree on $TV|_W$, then $\exists$ tubular nbhds U_0, U_1 of W and a diffeomorphism $\phi: U_0 \to U_1$ st $\phi^*\omega_1 = \omega_0$ and $\phi|_W = Id$.

Pf: The proof is similar to that of Darboux's theorem & the prop above. We define $\omega_t = (1-t)\omega_0 + t\omega_1$ which will be a symplectic form on a tubular nbhd of W.
As in the proof of the prop above, we can obtain a 1-form β on U st $d\beta = \dot{\omega}_t = \omega_1 - \omega_0$ and $\beta = 0$ on $TV|_W$.
Then X_t can be obtained by $\omega_t(X_t, -) = \beta$; $\beta = 0$ on pts in $W \Rightarrow X_t(w) = 0$ $\forall w \in W$.

$\Downarrow$ + W cpct

ϕ_t exists for $t \in [0,1]$ on some (tub) nbhd of W, U_0.

$\Rightarrow \exists \{\phi_t\}_{t\in[0,1]}$ on $U_0 \to V$ diffeo onto images st $\phi_t|_W = Id$

Take $\phi = \phi_1$, $U_1 = \phi_1(U_0)$.

Proposition: (Symplectic Normal Forms) Let ω_0, ω_1 be symplectic forms on a mfd V & $W \subset V$ a cpct submfd w/ $\omega_0|_{TW} = \omega_1|_{TW}$. Suppose $N := TW \cap (TW)^{\omega_0} = TW \cap (TW)^{\omega_1}$
$=: \ker(\omega_0|W) =: \ker(\omega_1|W)$
has constant rank, and $(TW^{\omega_0}/N, \omega_0) \cong (TW^{\omega_1}/N, \omega_1)$ as symplectic vbs.
Then $\exists$ tubular nbhds U_0, U_1 of W & a diffeomorphism $\phi: U_0 \to U_1$ st $\phi|_W = Id$ and $\phi^*\omega_1 = \omega_0$.

Pf: We use the following general isomorphism which exists whenever (E,ω) is a $2n$-rank sympl VB, $W \subset E$ a $(2k+l)$ rank subbundle such that $N := W \cap W^\omega$ has constant rank l.

$$(E,\omega) \overset{\text{sympl iso}}{\cong} (W/N, \omega) \oplus (W^\omega/N, \omega) \oplus (N \oplus N^*, \omega_{st})$$

(on fibers, $\omega_{st}((u,u^*),(v,v^*)) := v^*(u) - u^*(v)$)

Choosing a $J \in \mathcal{J}(E,\omega)$,
$$E = V_1 \oplus N \oplus V_2 \oplus V_3 \quad \text{where } V_1 = W \cap JW,\ V_2 = W^\omega \cap JW^\omega,\ V_3 = JN$$
& the iso can be obtained by
$$v_1 + n + v_2 + v_3 \longmapsto ([v_1], [v_2], (n, \omega(v_3, -)))$$

Here, $(E,\omega) = (TV|_W, \omega_0)$ and subbundle $= TW$ gives

$$(TV|_{W}, \omega_0) \cong (TW/N, \omega_0) \oplus (TW^{\omega_0}/N, \omega_0) \oplus (N \oplus N^*, \omega_{st})$$

& similarly $\quad \wr\wr$ as $\omega_0|_{TW} = \omega_1|_{TW} \quad \wr\wr$ by hypothesis $\quad ||$

$$(TV|_{W}, \omega_1) \cong (TW/N, \omega_1) \oplus (TW^{\omega_1}/N, \omega_1) \oplus (N \oplus N^*, \omega_{st})$$

where the vertical isos fix TW.

$\therefore \exists$ iso $\Psi : (TV|_W, \omega_0) \underset{\text{Sympl}}{\overset{\cong}{\longrightarrow}} (TV|_W, \omega_1)$ st $\Psi|_{TW} = Id$.

we can get from this a diffeomorphism of tubular nbhds $\psi : U_0 \to U_1$ of W st $\psi|_W = Id$ and $T\psi = \Psi$ (along W) $\Rightarrow \psi^* \omega_1 = \omega_0$ along W. (Use the exponential map wrt a Riem metric in normal directions)

Then we can apply the previous lemma to ω_0 & $\psi^* \omega_1$ to get another diffeomorphism between (possibly smaller) tubular nbhds of W $U_3, U_4 \subseteq U_0$

$\tilde{\phi} : U_3 \to U_4$ st $\tilde{\phi}^* (\psi^* \omega_1) = \omega_0$ & $\tilde{\phi}|_W = Id$

Note: we hadn't really required ω_0 & ω_1 to be defined on all of V here.

$\therefore \phi = \psi \circ \tilde{\phi}$ is the required diffeo.

This proposition immediately gives the normal forms (due to Weinstein) for isotropic, co-isotropic, sympl submanifolds.

Def A submanifold $W \subset (V, \omega)$ is <u>isotropic</u> if $T_w W \subset (T_w V, \omega_w)$ is an isotropic subspace $\forall w \in W$.

Similarly co-isotropic, symplectic, Lagrangian.

Cor: (Symplectic neighbourhood theorem). Let ω_0, ω_1, be symplectic forms on V mfd & $W \subset V$ a cpct submfd st $\omega_0|_{TW} = \omega_1|_{TW}$ are symplectic and $(TW^{\omega_0}, \omega_0) \overset{\text{sympl}}{\cong} (TW^{\omega_1}, \omega_1)$

Then $\exists$ tubular nbhds U_0, U_1 of W

st $\exists\, \phi : (U_0, \omega_0) \to (U_1, \omega_1)$ is a symplectomorphism fixing W.

Pf: $N = \{0\}$ as ω_0 (& ω_1) is a symplectic form on W.

Cor: (Isotropic nbhd thm) $W \subset V$ is a cpct submfd; ω_0, ω_1 sympl forms on V st $\omega_0|_W = \omega_1|_W = 0$ & $(TW^{\omega_0}/TW, \omega_0) \overset{\text{sympl}}{\cong} (TW^{\omega_1}/TW, \omega_1)$. Then $\exists$ tub nbhds of W U_0, U_1

& $\phi : U_0 \overset{\text{diffeo}}{\underset{\cong}{\longrightarrow}} U_1$ st $\phi|_W = Id$ & $\phi^* \omega_1 = \omega_0$

Pf: $N = TW$

Cor: (Coisotropic nbhd thm) As above, let ω_0, ω_1 be sympl forms on V, $W \overset{\text{cpt}}{\subset} V$ which is a coisotropic submfd wrt ω_0 & ω_1 $(\omega_0|_{TW} = \omega_1|_{TW})$

Then $\exists\, U_0, U_1, \phi$ as above.

Pf: $N = TW^{\omega_0}$, hence the other condition is not required.

Cor: (Lagrangian nbhd thm) Let $W \subset (V, \omega)$ be a compact Lagrangian submfd of (V, ω)

Then $\exists$ a tubular nbhd U of $W \subset T^* W$ & U' of $W \subset V$ and a diffeomorphism

$\phi : U \to U'$ st

1) $\phi|_W$ is the inclusion

& 2) $\phi^* \omega = \omega_{st}$ where ω_{st} is the standard symp str on $T^* W$.

Pf: Since W is Lagrangian, $TV|_W / TW \xrightarrow{\Phi} T^* W ; [v] \mapsto \omega(v, -)$ is well-defined & is an isomorphism.
Use this to get a diffeomorphism $\tilde{\phi} : U_1^{T^*W} \to U_2^{V}$ for tubular nbhds of W in $T^* W$ & V resp st $\tilde{\phi}|_W = Id$ & $\tilde{\phi}^* \omega = \omega_{st}$ on W.
(st Φ is the derivative of $\tilde{\phi}$ in normal direction, as in pf of prop above)
Then W is a coisotropic submfd of $T^* W$ wrt $\tilde{\phi}^* \omega$ & ω_{st} which agree on W. By the coisotropic nbhd thm,
$\exists\, \Psi : U_3 \to U_4$ (nbhds of W in $T^* W$) st $\Psi^* (\tilde{\phi}^* \omega) = \omega_{st}$. Take $\phi = \tilde{\phi} \circ \Psi$.

Contact manifolds

Let M be a smooth manifold of dim $(2n+1)$ and $\xi \subseteq TM$ a hyperplane field, ie a $(2n)$-rank subbndl of TM

Defn: ξ is a <u>contact structure</u> on M if for every local vector field $X \in \xi$ (ie $X(p) \in \xi$ $\forall p$ in the domain) there exists a local vector field $Y \in \xi$ st $[X,Y] \notin \xi$.
(M, ξ) is called a <u>contact manifold</u>.

Recall: Frobenius thm ξ is integrable $\iff$ $\forall$ local vf $X, Y \in \xi$, $[X,Y] \in \xi$.
$\left(d\alpha(X,Y) = \mathcal{L}_X(\alpha(Y)) - \mathcal{L}_Y(\alpha(X)) - \alpha([X,Y])\right)$ $\iff$ $d\alpha|_\xi \equiv 0$ for a 1-form α defined locally st $\ker \alpha = \xi$ there.

Whereas here, when ξ is a contact structure locally given by $\xi = \ker \alpha$ for a 1-form α, we have that $d\alpha|_\xi$ is a symplectic form on ξ (all locally).

We will assume that the contact structures we consider are globally $\ker \alpha$ for some $\alpha \in \Omega^1(M)$.

Rmk: $\xi = \ker \alpha$ is a contact structure $\iff$ $\alpha \wedge (d\alpha)^n \neq 0$ (nowhere zero) ie it is a volume form.

This implies that M has a natural orientation.

If $\alpha' \in \Omega^1(M)$ is st $\xi = \ker \alpha = \ker \alpha'$, then $\alpha' = f\alpha$ for some nowhere vanishing $f \in C^\infty(M)$,

and $d\alpha'|_\xi = f\, d\alpha|_\xi$ is also a symplectic form on M

Obs: Fixing $\alpha \in \Omega^1(M)$ w/ $\xi = \ker \alpha$,
$\exists$ a unique vector field $Y_\alpha \in \mathfrak{X}(M)$ st $d\alpha(Y_\alpha, -) = 0$ and $\alpha(Y_\alpha) = 1$.
This is called the Reeb vector field determined by α.

Cartan's magic formula $\Rightarrow$ $\mathcal{L}_{Y_\alpha} \alpha = \underbrace{d(\overset{1}{\alpha(Y_\alpha)})}_{=0} + \underbrace{d\alpha(Y_\alpha, -)}_{=0 \text{ by def}} = 0$

$\Rightarrow (\phi^t_{Y_\alpha})^* \alpha = \alpha$ where $\phi^t_{Y_\alpha}$ is the flow of Y_α.

$\Rightarrow \phi^t_{Y_\alpha}$ preserves ξ

Rmk: Y_α depends on α & is not intrinsic to (M, ξ).

Defn: A diffeomorphism $f: (M_1, \xi_1) \to (M_2, \xi_2)$ ("contact mfds") is called a <u>contactomorphism</u> if $f^* \xi_2 = \xi_1$

Ex: → $(\mathbb{R}^{2n+1}, \xi_{std})$ where $\xi_{std} = \ker \alpha_{std}$, $\alpha_{std} = dz - \sum_{j=1}^n y_j\, dx_j$
$(x_1, \dots, x_n, y_1, \dots, y_n, z)$

More generally,
→ For any exact (ie sympl form is exact) symplectic manifold $(W, \omega = d\alpha)$, $\lambda = dt - \alpha$ is a contact form on $W \times \mathbb{R}$ where t is the $\mathbb{R}$-coordinate.

$\therefore (W \times \mathbb{R}, \ker \lambda)$ is a contact manifold.

The Reeb vector field corresponding to λ is ∂_t.

→ Suppose (V, J) is an almost complex manifold and $\Sigma \subset V$ a coorientable hypersurface. ($\dim V = 2n$)

Define a $(2n-2)$-plane distribution on Σ by $\xi_p := T_p\Sigma \cap J(T_p\Sigma)$,
"field of complex tangencies"

the maximal complex subspace of $T_p\Sigma$.

Suppose Σ is cooriented by a transverse vector field ν st $J\nu \subset T\Sigma$.

Then $\xi = \ker \alpha$ for some $\alpha \in \Omega^1(\Sigma)$ st $\alpha(J\nu) > 0$.

$\omega_\Sigma := d\alpha|_\xi$ is independent of α upto multiplication by a positive function

Defn: Σ is said to be <u>J-convex</u> if $\omega_\Sigma(X, JX) > 0 \ \forall X \neq 0 \in \xi$.

Then (Σ, ξ) is a contact manifold.

Conversely, given any cooriented contact manifold (Σ, ξ), we can choose an acs J' on ξ st (exists as $\xi \to \Sigma$ is a sympl VB.)

$d\alpha(-, J'-)$ defines a bundle metric on ξ ($= \ker\alpha$), and extend J' to an acs J on $\underbrace{\Sigma \times (-\varepsilon, \varepsilon)}_{V}$

Σ is then a J-convex hypersurface of V.

Defn: Let $(M, \xi = \ker \alpha)$ be a contact mfd of dimension $2n+1$.

An immersion $\phi : \Lambda \to M$ is called <u>isotropic</u> if $T\phi(T\Lambda) \subset \xi$.

Rmk: $T\phi(T_q\Lambda) \subset \underbrace{(\xi_{\phi(q)}, d\alpha_{\phi(q)})}_{\text{sympl vs}}$ is then an isotropic subspace.

For locally defined vector fields X, Y in Λ, $d\alpha(T\phi(X), T\phi(Y))$

$$= d(\phi^*\alpha)(X,Y) = \mathcal{L}_X \overbrace{(\underbrace{\phi^*\alpha(Y)}_{\equiv 0})}^{0} - \mathcal{L}_Y \overbrace{(\underbrace{(\phi^*\alpha)(X)}_{\equiv 0})}^{0} - \overbrace{\phi^*\alpha[\underbrace{X,Y}_{\in T\Lambda}]}^{0}$$

$$= 0$$

$\therefore \dim \Lambda \leq n$ if ϕ is an isotropic immersion into a manifold of dim $2n+1$,

Defn: If $\dim \Lambda = n$ & $\phi : \Lambda \to (M, \xi)$ is an isotropic immersion.

ϕ is called a <u>Legendrian immersion</u>.

EX: Consider the contact manifold $J^1(L) = T^*L \times \mathbb{R}$ w/ $\xi := \ker(dt - \lambda)$ where λ is the canonical 1-form on T^*L.

(from the previous ex) $\lambda = p\,dq$ in local coords.

Denoting elements of $J^1(L)$ by (q, p, z) where $q \in L$, $p \in T^*_qL$, $z \in \mathbb{R}$

any $f : L \to \mathbb{R}$ defines a section of $J^1L \to L$ by $L \xrightarrow{\hat{f}} J^1L$, $q \mapsto (q, df(q), f(q))$.

Then the image $\hat{f}(L)$ is a submanifold of J^1L

and $(\hat{f})^*(dz - p\,dq)(X) = dz(d\hat{f}(X)) - \underbrace{df(q)}_{\text{loc coord } p \text{ at } \hat{f}(q)} \underbrace{dq(d\hat{f}(X))}_{X}$

$$= df(X) - df(X) = 0$$

$\therefore \hat{f}(L) \subset (J^1L, \xi)$ is a Legendrian submanifold.

There is an analogue of Moser's Stability theorem for contact mfds.

Thm: (Gray's Stability Thm) Let $\{\xi_t\}_{t\in[0,1]}$ be a smooth family of contact structures on a closed manifold M. Then $\exists$ a diffeotopy $\{\phi_t : M\to M\}_{t\in[0,1]}$ w/ $\phi_0 = Id$ & $\phi_t^*\xi_t = \xi_0$ $\forall t\in[0,1]$.

This is proved by a variation of Moser's Deformation trick.

And the Contact Darboux's theorem also can be proved by that method -

Thm: (Contact Darboux's theorem) A contact manifold (M,ξ) is locally contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{std})$ $\dim M = 2n+1$.

Notn: Let $(M,\xi_M), (N,\xi_N)$ be contact mfds w/ $\dim M \le \dim N$

Defn: An immersion $f: M\to N$ is called isocontact if $f^*\xi_N = \xi_M$.

A monomorphism (fiberwise injective) $F: TM \to TN$ is isocontact if $F^*\xi_N = \xi_M$ and

$F:\xi_M\to\xi_N$ is conformally symplectic,

ie. $(F)^* d\alpha_N = g\, d\alpha_M$ for a positive function g & $\ker(\alpha_M) = \xi_M$, $\ker(\alpha_N) = \xi_N$.

Thm: (Contact isotropic neighbourhood theorem) If M,N are as above and $\Lambda\subset M$ is an isotropic submanifold w/ an isotropic immersion $f:\Lambda\to N$ covered by an isocontact monomorphism $F: TM \to TN$, $(Tf = F|_{T\Lambda})$ then,

$\exists$ a neighbourhood $U\subset M$ of Λ and an isocontact immersion $g: U\to N$ st $g|_\Lambda = f$ and $dg|_\Lambda = F|_\Lambda$. Also, g can be chosen to be an embedding if f is.

Cor: A neighbourhood of a Legendrian submanifold $\Lambda\subset (M,\xi_M)$ is contactomorphic to a neighbourhood of the zero section in $J^1\Lambda$.

Real analytic approximations of isotropic submanifolds

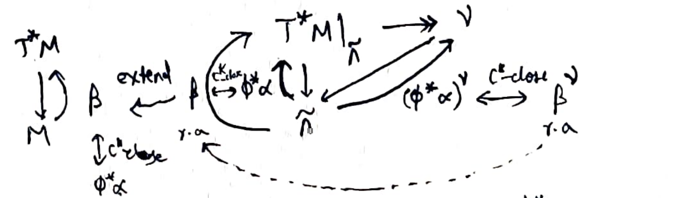
Thm: Let Λ be a closed isotropic C^k-submanifold ($k \geq 1$) in a real analytic closed contact mfd $(M, \ker\alpha)$ where M and $\alpha \in \Omega^1(M)$ are both real analytic.
Then, there exists a real analytic isotropic submanifold $\Lambda' \subset (M, \alpha)$ arbitrarily C^k-close to Λ.
Moreover, any C^k-isotopy $\{\Lambda_t\}_{t\in[0,1]}$ of closed isotropic C^k-submanifolds of (M,α) st Λ_0 & Λ_1 are real analytic can be modified to a real analytic isotopy $\{\Lambda'_t\}_{t\in[0,1]}$ of real analytic isotropic submanifolds Λ'_t st Λ'_t is arb. C^k-close to Λ_t & $\Lambda_0 = \Lambda'_0$, $\Lambda_1 = \Lambda'_1$

The proof uses Thm [5.53] from the book which we recall from the first talk.

Thm [5.53]: Suppose $E \to M$ is a r.a fiber bundle, $N \subset M$ properly embedded r.a submfd, $k \geq d \geq 0$ integers, $h: M \to \mathbb{R}_+$ a cts function. If $f: M \to E$ is a smooth section st its normal jet $J^d f$ along N is r.a, then f can be approximated by a r.a section F: $\exists F: M \to E$ section, r.a, st $j^d f = j^d F$ along N & $|D^k F(x) - D^k f(x)| < h(x)$ $\forall x \in M$
↑ includes derivatives of lower order too.

Pf: Since M is a r.a mfd & $\Lambda \overset{C^k\text{-submfd}}{\subset} M$, $\exists \tilde\Lambda \subset M$, a r.a submfd C^k-close to Λ, (ie $i: \Lambda \hookrightarrow M$ & $\tilde i: \tilde\Lambda \hookrightarrow M$ are C^k-close) ↳ not necessarily isotropic
and $\exists \phi: M \xrightarrow[\cong]{C^k} M$ diffeo st $\phi(\tilde\Lambda) = \Lambda$ & ϕ is C^k-close to Id. [5.48]
$\phi^*\alpha \in \Omega^1(M)$, so it is a section of $T^*M \to M$. r.a v.b as M is r.a.
① $\to \phi^*\alpha|_{T\tilde\Lambda} = 0$ as Λ is isotropic.
Restricting it to $\tilde\Lambda$ gives a section of $T^*M|_{\tilde\Lambda} \to \tilde\Lambda$.
Let $\nu \to \tilde\Lambda$ be the normal bundle to the zero section $T\tilde\Lambda \subset T^*M|_{\tilde\Lambda}$ wrt a r.a metric.
Then quotienting by $T\tilde\Lambda$ produces a C^k-section $(\phi^*\alpha)^\nu$ in ν (well-defined by ①)

By [5.49] (r.a approx. sections)
$\exists \beta^\nu$ a section of ν that is C^k-close to $(\phi^*\alpha)^\nu$. Extend β^ν to a section β of $T^*M|_{\tilde\Lambda}$ st $\beta|_{T\tilde\Lambda} = 0$ & β C^k-close to $\phi^*\alpha$ along $\tilde\Lambda$. Then extend β to a 1-form on M st β is C^k-close to $\phi^*\alpha$.



(β is r.a along $\tilde\Lambda$), we get that
Applying thm [5.53] w/ $E = T^*M$, $d = 0$, $f = \beta$, & $\beta|_{T\tilde\Lambda} = \tilde\alpha|_{T\tilde\Lambda} = 0$
$\exists$ a r.a 1-form $\tilde\alpha$ that is C^k-close to β,
Since β is C^k-close to $\phi^*\alpha$ & ϕ is C^k-close to Id & $\tilde\alpha$ is C^k-close to β, $\tilde\alpha$ is C^k-close to α.

Define $\alpha_t := (1-t)\tilde\alpha + t\alpha$, which is a real analytic htpy of r.a. contact forms.
Gray's stability thm $\Rightarrow$ $\exists$ diffeotopy $\phi_t: M \to M$ st $\phi_t^* \xi_t = \xi_0$ where $\xi_t = \ker\alpha_t$,
or equivalently $\phi_t^*\alpha_t = f_t\tilde\alpha$ for some family of positive functions f_t.
Since the ϕ_ts can be constructed as solutions to an ODE whose coefficients are C^k-small (as $\dot\alpha_t = \alpha - \tilde\alpha$ is C^k-close to 0), ϕ_t are all real analytic [Rmk 5.39], and close to Id, and depend real-analytically on t.
$\therefore \Lambda' := \phi_1(\tilde\Lambda)$ is real-analytic, close to Λ and $\alpha|_{T\Lambda'} = 0$.

References:

1) From Stein to Weinstein and Back : Symplectic Geometry of Affine Complex Manifolds

- Kai Cieliebak , Yakov Eliashberg

2) Lectures on Holomorphic Curves in Symplectic and Contact Geometry

- Chris Wendl

3) Differential Geometry - 1 lecture notes

- Chris Wendl

[Ban] 4) Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique

- Augustin Banyaga

5) Introduction to Symplectic Topology

- Dusa McDuff , Dietmar Salamon