The Weinstein and Stein existence theorems

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We have developed a rich theory of Stein and Weinstein manifolds. We have also introduced some simple examples of Stein and Weinstein manifolds, like $\mathbb{R}^{2n}$, Kähler manifolds or punctured Riemann surfaces. However, it would be useful, if we had even more examples. Since stable discs in Weinstein manifolds are isotropic, the critical points must have index $\leq n$. The best we could hope for is a statement that upgrades any such Morse manifold into a Weinstein manifold. In this talk I present a theorem that does exactly that. My second big theorem then upgrades a Weinstein manifold to a Stein manifold, thereby also proving the existence of many Stein manifolds.

1 Background

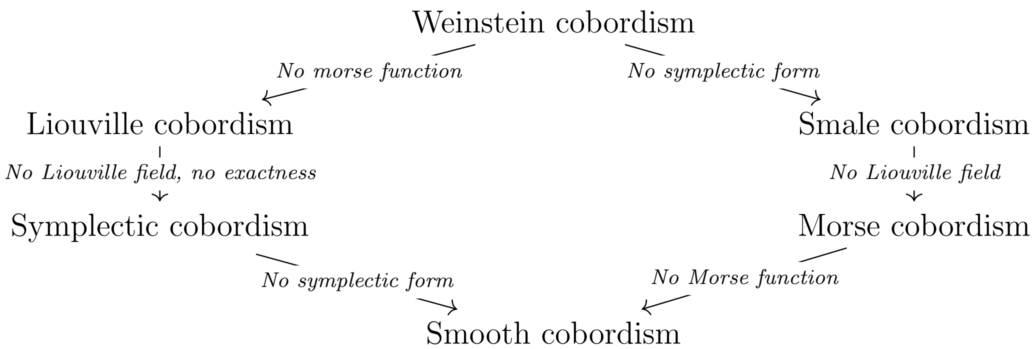
Definition 1.1. *A* <u>*Weinstein manifold*</u> *is a tuple* (V, ψ, ω, X) *where:*

- ψ *is an exhausting Morse function*
- ω *is an exact symplectic form* $\omega = d\sigma$
- X *is a gradient-like Liouville vector field*

Definition 1.2. *A* <u>*Weinstein cobordism*</u> (W, ψ, ω, X) *is a Weinstein manifold that is also a cobordism, and*

- ψ *is constant on the two boundaries*
- X *points inward on* $\partial_- W$ *and outward on* $\partial_+ W$*.*

We can weaken certain parts of the definitions, to obtain different objects. We repeat the statements that will be required for this talk.



2 Weinstein existence theorem

Theorem 2.1 (Weinstein existence theorem)**.** *For* $n > 2$*, let* (V, ψ) *be a* $2n$*-dimensional smooth manifold with an exhausting Morse function* ψ *whose critical points are of index* $\leq n$*. Let* ω *be a non-degenerate (not necessarily closed)* 2*-form on* V*. Then* ω *can be isotoped into a Weinstein structure* (ω, X, ψ) *on* V *with fixed* ϕ*. Moreover, we can arrange that* (ω, X, ϕ) *is flexible.*

Flexible means that a Lagrangian n-dimensional stable disc is loose in V, i.e. that it fulfils some h-principle. In this talk we will not go into flexibility. We will also not describe the 4-dimensional case in details as this requires extra care. Remember that any Weinstein manifold is a Weinstein domain with cylinders attached. Therefore, the Weinstein existence theorem is a direct consequence of:

Theorem 2.2 (Weinstein existence theorem for cobordisms)**.** *For $n > 2$, let (W, ϕ) be a $2n$-dimensional Morse cobordism whose critical points are of index $\leq n$. Let ω be a non-degenerate (not-necessarily closed) 2-form and let X be any gradient-like vector field on W such that $(W, \omega, X, \psi)|_{\mathcal{O}p\partial_- W}$ is Weinstein. Then there exists an isotopy of (ω, X) outside $\mathcal{O}p\partial_- W$ s.t. (W, ω, X, ϕ) becomes Weinstein.*

If $n = 2$, then we also require the contact structure defined by the Liouville form $\lambda = \imath_Y \omega$ on $\partial_- W$ to be overtwisted. Moreover, we can arrange that (ω, X, ϕ) is flexible.

The last theorem allows us to choose any Weinstein structure on the lower boundary and extend this to a Weinstein structure on the entire cobordism. Note that because every cobordism is composed of elementary cobordisms, we only need to show the statement for the elementary case. For simplicity, we assume that the cobordism has only one critical point p of index $\leq n$. The proof outlines as follows:

1. Homotope (ω, X) thereby upgrading the stable disc Δ of p to be η-isotropic.

2. Homotope (ω, X) thereby upgrading W to be Liouville around Δ and $\partial_- W$.

3. Homotope (ω, X) thereby upgrading W to be Liouville everywhere.

4. The resulting cobordism will also be Weinstein.

At the end of each paragraph, I will include the necessary statements from previous chapters. We start with the following theorem. It states that we can find an isotopy that moves any disc that is transversely attached to the boundary $\partial_- W$ into a totally real subspace and J-orthogonal position. Note that this theorem is the reason for the overtwistedness assumption.

Proof. **Step 1.** Let $\Delta : D^k \hookrightarrow W$ be the stable manifold of p. Since Δ runs along X and X points inward at $\partial_- W$, the stable disc is attached transversely to $\partial_- W$. Now we choose an auxiliary almost complex structure J on W. Theorem 7.34 gives us an isotopy $f_t : D^k \hookrightarrow W$ of Δ, so that $f_0 = \Delta$ and $f_1(D^k)$ is totally real in W and J-orthogonally attached to $\partial_- W$. Write $\Delta' = f_1$ going forward. We extend f_t to an ambient isotopy $f_t : W \to W$.

Theorem 2.3 (Theorem 7.34, h-principle)**.** *For $n > 2$ suppose that (W, J) is an almost complex manifold of real dimension $2n$ and $U \subseteq W$ is a domain with smooth J-convex boundary. Suppose that an embedding $f_0 : D^k \hookrightarrow W, k \neq n$ transversely attaches D^k to U along ∂D^k. Then there exists an isotopy $f_t : D^k \hookrightarrow W, t \in [0, 1]$, through embeddings transversely attaching D^k to U, such that and f_1 is totally real and J-orthogonal to ∂U.*

If $k = n = 2$, we assume in addition, that the induced contact structure on ∂U is overtwisted.

In the case $k = n > 2$ we can arrange that the Legendrian embedding $f_1|_{D^k} : \partial D^k \hookrightarrow \partial W$ is loose, while for $k = n = 2$ we can arrange that the complement $\partial U \backslash f_t(\partial D^2)$ is overtwisted for all $t \in [0, 1]$.

Making Δ' totally real is the first step in making it an isotropic subspace. However, notice that due to the isotopy Δ' might now be disconnected from its critical point p. So instead of moving Δ forward along the isotopy, we move J backward along the ambient isotopy. By symmetry, Δ is now f_1^*J-totally real and f_1^*J-orthogonally attached to $\partial_- W$. (Note that in the book the homotopies are still illustrated by how they act on Δ, even though we're not really interested in moving Δ). However, now we face a new problem: f_1^*J is no longer compatible with ω. The solution to this is to also move ω, but we need to be careful since we're not allowed to touch ω on $\mathcal{O}p\partial_- W$. So, we define a new homotopy g_t from modification of f to remedy this fact. Take some regular value $c + \varepsilon > c = \phi^{-1}(\partial_- W)$, define $U = \{f(x) < c + \varepsilon\}$ and identify the level sets of $[c, c + \varepsilon]$ via J. We define g_t as the homotopy, that is the identity for the level sets $[c, c + \frac{\varepsilon}{2}]$, $g_t = f(t)$ for the level set $c + \varepsilon$ and smoothly interpolates between the other level sets by stopping the homotopy early. Then $k_t = f_t^{-1} \circ g_t$ has the following properties:

- k_t equals the identity on $W \backslash U$ and f_t^{-1} on $\mathcal{O}p\partial_- W$
- Preserves ϕ everywhere
- $X_t = k_t^* X = X$ on $W \backslash U$ and $\mathcal{O}p(\partial_- W)$ and X_t is everywhere gradient-like for $\phi_t = k_t^* \phi$.
- $\omega_t = g_t^* \omega$ is compatible with J_t.
- $k_1^{-1}(\Delta)$ is J_1-orthogonally real and J_1-orthogonally attached to $\partial_- W$

Therefore, Δ is isotropic with respect to ω_1.

Step 2. Apply Lemma 12.16. Now (W, ϕ, ω, X) is a Liouville manifold in a neighbourhood of $\Delta \cup \partial_-(W)$

Lemma 2.4 (Lemma 12.16)**.** *Let* (W, X_0, ϕ) *be an elementary Smale cobordism and* ω_0 *a nondegenerate* 2*-form on* W*. Let* Δ *be the stable disc of* ϕ*. Suppose that* Δ *is* ω_0*-isotropic and the pair* (ω_0, X_0) *is Liouville on* $\mathcal{O}p(\partial_- W)$*. Then for any neighbourhood* U *of* $\partial_- W \cup \Delta$ *there exists a homotopy* (ω_t, X_t) *with the following properties:*

1. X_t *is a gradient-like vector field for* ϕ *and* ω_t *is a nondegenerate* 2*-form on* W *for all* $t \in [0, 1]$
2. $(\omega_t, X_t) = (\omega_0, X_0)$ *outside* U *and on* $\Delta \cup \mathcal{O}p(\partial_- W)$
3. (ω_1, X_1) *is a Liouville structure on* $\mathcal{O}p(\partial_- W \cup \Delta)$*.*

Next we extend the Liouville structure in a neighbourhood of $\Delta \cup \partial_-(W)$ to the entire cobordism. The trick is to follow the vector field X.

Step 3. We apply Proposition 9.19. This gives us a diffeotopy $h_t : W \to W$ that flows the W into the $\mathcal{O}p(\Delta \cup \partial_- W)$. The pullback $(h_t^* \omega, h_t^* X)$ gives a Liouville structure on all of W. The proposition ensures that ψ stays gradient-like. Therefore $(W, h_t^*\omega, h_t^* X, \phi)$ is Weinstein.

Proposition 2.5 (Proposition 9.19)**.** *Let* (W, X, ϕ, Δ) *be as above. Let* $a_- = \phi^{-1}(\partial_- W)$*. Fix an open neighbourhood* U *of* $\partial_- W$ *and a regular value* $c > a_-$*. Then there exists a diffeotopy* $h_t : W \to W$ *with the following properties:*

- $h_0 = Id$ *and* $h_t = Id$ *on* $\mathcal{O}p(\partial W \cup \Delta)$
- h_t *preserves trajectories of* X
- $h_1(\{\phi \leq c\}) \subseteq U$

In particular, the map $\phi_t := \phi \circ h_t^{-1}$ *stays gradient-like w.r.t.* X*.*

□

3 Existence of Stein manifolds

In this part we show how to upgrade a Weinstein structure to a Stein structure. We start similarly to the last section with the existence theorem for manifolds.

Theorem 3.1 (Stein existence theorem)**.** *Let* $\mathfrak{W} = (V, \omega, X, \phi)$ *be a Weinstein manifold. Then there exists a Stein structure* (J, ϕ) *on* W *such that the Weinstein structures* $\mathfrak{W}$ *and* $\mathfrak{W}(J, \phi)$ *are homotopic with fixed function* ϕ*.*

This follows non-trivially from the Stein existence theorem for cobordisms. The proof uses induction over sublevel sets.

Theorem 3.2 (Stein existence theorem for cobordisms)**.** *Let* $\mathfrak{W} = (W, \omega, X, \psi)$ *be a Weinstein cobordism which is Stein on* $\mathcal{O}p\partial\_W$*. Then after target reparametrising, the Stein structure extends to a Stein structure* (J, ϕ) *on* W *such that the Weinstein structures* $\mathfrak{W}$ *and* $\mathfrak{W}(J, \phi)$ *are homotopic with fixed* ϕ*.*

To finish in time, I will present the next two results before starting the proof.

Theorem 3.3 (Parametric Stein existence theorem for cobordisms)**.** *Let* $\mathfrak{W}_u = (W, \omega_u, X_u, \phi)$*,* $u \in D^k, k \geq 0$*, be a family of Weinstein cobordism structures which share the same Morse function* ϕ*. Suppose all* $\mathfrak{W}_u$ *are Stein near* ∂_W *and for all* $u \in \partial D^k$*,* $\mathfrak{W}_u$ *are Stein on* W*. Then after target reparametrizing* ϕ*, there exists a family of Stein structures* (J_u, ϕ) *for all* $u \in D^k$ *and a family of Weinstein homotopies* $\mathfrak{W}_{t,u}$ *such that*

- $\mathfrak{W}_{0,u} = \mathfrak{W}(J_u, \phi)$ *and* $\mathfrak{W}_{1,u} = \mathfrak{W}_u$ *for all* $u \in D^k$
- $\mathfrak{W}_{t,u} = \mathfrak{W}_u$ *near* ∂_W
- $\mathfrak{W}_{t,u} = \mathfrak{W}_u$ *for* $u \in \partial D^k$

Corollary 3.3.1. *The map* $\mathfrak{W} :$ ***Stein***$(W, \phi) \to$ ***Weinstein***(W, ϕ) *is a weak homotopy equivalence.*

The proof of the Stein existence theorem requires the existence of integrable complex structures.

Theorem 3.4 (Theorem 8.11, Existence of complex structures, Gromov, Landweber)**.** *Let* (W, J, ϕ) *be a* $2n$*-dimensional almost complex Morse cobordism, where* ϕ *has no critical points of index* $> n$*. Let* Δ *be the skeleton of* ϕ *with respect to some gradient-like* X*. Then* J *can be* C^0*-approximated by an almost complex structure which coincides with* J *outside a neighbourhood of* L *and is integrable on* $\mathcal{O}p(\Delta \cup \partial\_W)$*. In particular,* J *is homotopic to an integrable complex structure via a homotopy fixed on* $\mathcal{O}p\partial\_W$*.*

The proof of the Stein existence theorem is a consequence of the following proposition. It states that there is a homotopy fixing a neighbourhood of the lower boundary that extends the Stein structure to the handle. Note that this does change the Morse function ϕ but in a way that can be remedied later on:

Proposition 3.5. *Let $\mathfrak{W}_0 = \mathfrak{W} = (W, \omega, X, \psi)$ be a Weinstein cobordism and J an integrable complex structure on W. Suppose that on $\mathcal{O}p\partial_-W$ the function $\phi = \phi_0$ is J-convex and $\mathfrak{W}$ coincides with $\mathfrak{W}(J, \phi)$. Suppose moreover that J is homotopic rel ∂_-W to an almost complex structure compatible with ω. Then there exists a homotopy of Weinstein structures $\mathfrak{W}_t = (\omega_t, X_t, \phi_t)$ on W, and a regular value c of the function ϕ_1 with the following properties:*

1. *$\mathfrak{W}_t$ agrees with $\mathfrak{W}_0$ on $\mathcal{O}p\partial_-W$ and up to scaling on $\mathcal{O}p\partial_+W$.*
2. *on $W' = \{\phi_1 \leq c\}$ the function ϕ_1 is J-convex and $\mathfrak{W}_1|_{W'} = \mathfrak{W}(J, \phi_1)$*
3. *on $\{\phi_1 \geq c\}$ the function ϕ_1 has no critical points.*
4. *$\phi_t = \phi \circ f_t$ for a diffeotopy $f_t : W \to W$ fixed on $\mathcal{O}p\partial W$ with $f_0 = Id$.*

Just as in the Weinstein existence theorem, we only need to show this for elementary cobordisms. We will further assume that they have only one critical point. The proof is again divided into four steps:

1. Homotope (W, ω, X, ϕ) thereby upgrading ϕ to be J-convex around p. (We will not tough the neighbourhood of p again)
2. Homotope J thereby making it compatible with ω and the stable disc Δ totally real and J-orthogonally attached to ∂_-W.
3. Homotope the Weinstein structure (including ϕ) thereby making ϕ at the handle J-convex and the handle Stein
4. Homotope ϕ to give it the required properties 1.-3. Make sure that the cobordism is still Weinstein. Apply Lemma 9.38 for property 4.

Proof. **Step 1.** Using Corollary 12.13 we deform $\mathfrak{W}$ so that ϕ is J-convex and $\mathfrak{W} = \mathfrak{W}(J, \phi)$ around p. (We won't change the neighbourhood of p ever again)

Corollary 3.5.1 (Corollary 12.13)**.** *A Weinstein structure with hyperbolic an embryonic critical points is homotopic to one which is Stein for a given complex structure near the critical points.*

Step 2. Just like before, choose a homotopy J_t rel $\mathcal{O}p(\partial W_- \cup p)$ of almost complex structures such that $J_1 = J$ and J_0 is compatible with ω. By Corollary 7.31 there is a isotopy of discs Δ_0 that are J_t-totally real and fixed on $\Delta \cap \mathcal{O}p(\partial_- W \cup p)$. We extend this isotopy to a global diffeotopy $g_t : W \to W$ rel $\mathcal{O}p(\partial_- W \cup p)$. As this would again pull the stable disc away from p, instead of moving back the disc, we move forward the Weinstein structure, i.e. we replace $\mathfrak{W}$ by $(g_1)_* \mathfrak{W}$. Δ is now totally real and J-orthogonally attached.

Corollary 3.5.2 (Corollary 7.31, h-principle). *Let (W, J_0) be an almost complex manifold and $f_0 : \Delta \hookrightarrow W$ be a totally real embedding. Let J_t be a homotopy of almost complex structures. Let $f_t|_{\mathcal{O}pB}$ be an isotopy of J_t-totally real embeddings into the neighbourhood of a closed $B \subseteq \Delta$. Then f_t can be extended to a J_t-totally real embedding $f_t : \Delta \to W$.*

Step 3. Remember that J is actually complex (not just almost). So we can apply Lemma 8.7. This gives us a J-convex function $\tilde{\phi}$ s.t. Δ is J-orthogonal to the level sets of $\tilde{\phi}$. This Weinstein structure $\mathfrak{W}(J, \tilde{\phi})|_{\mathcal{O}p(\partial_- W \cup \Delta)}$ is Stein on the handle (and $\tilde{\phi}$ is undefined outside of the handle). Note however, that Lemma 8.7 does not give an isotopy and $\mathfrak{W}(J, \tilde{\phi})$ is only defined on the handle. So next, we apply Proposition 12.14. We obtain a homotopy of the original Weinstrein structure to a Weinstein structure defined on W that is Stein on the handle.

Lemma 3.6 (Lemma 8.7). *Let (W, J) be a complex manifold, $\Delta \subseteq W$ a compact totally real submanifold, $\phi : W \to \mathbb{R}$ a smooth function, and X a nowhere vanishing vector field which is tangent to Δ and gradient-like for ϕ. Let $K \subseteq \Delta$ be a compact subset such that on $\mathcal{O}pK \subseteq W$ the function ϕ is J-convex, $\nabla_\phi \phi = X$, and $\Delta \cap \mathcal{O}pK$ is J-orthogonal to the level sets of ϕ. Then there exists a J-convex function $\tilde{\phi}$ on $\mathcal{O}p\Delta$ which agrees with ϕ on $\Delta \cup \mathcal{O}pK$ such that Δ is J-orthogonal to the level sets of $\tilde{\phi}$, and $\nabla_{\tilde{\phi}} \tilde{\phi} = X$ along Δ.*

Proposition 3.7 (Proposition 12.14). *Given a Weinstein manifold $(W, \omega_0, X_0, \phi_0)$ with a non-degenerate critical point p of index k and an embedded k-disc $\Delta \subseteq W_p^-$ containing p. Let $(\omega_{loc}, X_{loc}, \phi_{loc})$ be a Weinstein structure on a neighbourhood W_{loc} of Δ which coincides with (ω_0, X_0, ϕ_0) on $\Delta \cup \mathcal{O}p\partial\Delta$. Then there exists a homotopy of Weinstein structures (ω_t, X_t, ϕ_t) on W such that $(\omega_t, X_t, \phi_t) = (\omega, X, \phi)$ outside W_{loc} and on $\Delta \cup \mathcal{O}p\partial\Delta$. The* *homotopy ends with* $(\omega_1, X_1, \phi_1) = (\omega_{loc}, X_{loc}, \phi_{loc})$ *on* $\mathcal{O}p\Delta$.

Step 4. Now, we apply the homotopy to get the properties 1.-4. We use Theorem 8.5 and get a deformation as described in the theorem. Cieliebak and Eliashberg get a slightly different deformation than described in the Lemma, but the proof that this deformation exists should be the same:

- $\phi_0 = \phi|_U$
- $\phi_t = g_t\phi_0$ for some target equivalence g_t near ∂U and equal to ϕ near $\partial_- W \cup \Delta$
- ϕ_t has no critical points besides p
- some level set $\{\phi_1 = c\}$ surrounds $\partial_- W \cup \Delta$ in U

The difference between these two, is that in one case the level sets have been pulled up and in the other case they have been pulled down by a target equivalence g_t. We'll assume that g_t is nicely behaved (s.t. the contact structure on $\partial_+ W$ is preserved later on). Remember that the Liouville form of the Stein manifold is defined as $\lambda_t = -d\phi_t \circ J = (g_t' \circ \phi)(d\phi \circ J) = f_t$ with $f_t := g_t' \circ \phi$, so this satisfies

$$f_t + df_t(X) = g_t' \circ \phi + (g_t'' \circ \phi)(d\phi(X)) > 0$$

so according to Lemma 12.1 the Weinstein structure on U extends to a Weinstein structure $\mathfrak{W}_t$ on W.

Theorem 3.8 (Theorem 8.5)**.** *Let* (W, J) *be a complex manifold with a compact* J*-concave boundary* $\partial_- W$*. Let* $\phi_0 : W \to \mathbb{R}$ *be a* J*-convex Morse function which is constant on* $\partial_- W$ *and has no critical points on* $\partial_- W$*. Let* p *be a critical point of* ϕ_0 *and denote by* $\Delta \subseteq W$ *the stable disc of* p*. Then for any neighbourhood* $U \supseteq \partial_- W \cup \Delta$ *there exists a homotopy of* J*-lc functions* $\phi_t : W \to \mathbb{R}$ *with the following properties:*

1. ϕ_t *is equal to* ϕ_0 *near* $\partial_- W$ *and near* ∂U*, and target equivalent to* ϕ_0 *near* Δ*.*
2. $\phi_t|_N$ *has the unique critical point* p *with stable disc* Δ
3. *some level set* $\{\phi_1 = c\}$ *surrounds* $\partial_- W \cup \Delta$ *in* U*.*

Lemma 3.9 (Lemma 12.1). *Let (W, ω, X, ϕ) be a Weinstein cobordism with Liouville form λ. Then for a function $f : W \to \mathbb{R}$ the following holds: The 1-form $f\lambda$ defines a Weinstein structure iff $f > 0$ and $k := f + df(X) > 0$.*

□

Proof of Stein existence theorem. In the previous proposition applied a homotopy to ϕ which shouldn't have been allowed. To fix this, we apply the homotopy to the complex structure J instead.

Let $\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t = \phi_t \circ f_t)$ and W' be as in the proposition. Let $g_t : W \hookrightarrow W$ be the isotopy that flows W along the trajectories of X such that $g_1(W) = W'$. Set $h_t = g_t \circ f_t^{-1}$. Then $(h_t)_*\phi$ stays fixed (up to target equivalence) and $\mathfrak{W}((h_t)_*J, (h_t)_*\phi)$ is a Weinstein homotopy with fixed ϕ. □

Theorem 3.10 (Stein existence theorem). *Let $\mathfrak{W} = (W, \omega, X, \psi)$ be a Weinstein cobordism and J an integrable complex structure on W. Suppose that on $\mathcal{O}p\partial\_W$ the function ϕ is J-convex and $\mathfrak{W}$ coincides with $\mathfrak{W}(J, \phi)$. Suppose moreover that J is homotopic rel ∂_W to an almost complex structure compatible with ω. Then (up to target reparametrisation) there exists an isotopy $h_t : W \hookrightarrow W$ rel $\mathcal{O}p\partial\_W$ with $h_0 = Id$ such that the function $h_{1*}\psi$ is J-convex and the Weinstein structures $\mathfrak{W}(h_1^*J, \phi)$ and $\mathfrak{W}$ on W are homotopic rel $\mathcal{O}p\partial\_W$ with fixed function ϕ.*