Stein and Weinstein Deformations.

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§1 : Big Picture:

Let W be a compact smooth mfd with bdry.

We've considered two main structures on W

- Weinstein (W, ω, X, ϕ)

Liouville $\mathcal{L}_X \omega = \omega$

symplectic

exhausting generalized Morse fn.

Weinstein "domain"

- Stein (W, J, ϕ)

almost complex str.

J-convex

$\rightarrow$ $\omega_\phi = -dd^{\mathbb{C}} J$ symplectic

$g_\phi(v, w) := \omega_\phi(v, Jw)$ is Riem. metric.

can define $X_\phi := \nabla_{g_\phi} \phi$

to yield Weinstein structure.

Thus, we have a natural map

$$\underline{\underline{\mathrm{Stein}}} \longrightarrow \underline{\underline{\mathrm{Weinstein}}}$$

Now how do we go the other direction?

In what sense does a Weinstein structure yield a Stein structure?

Let $\underline{\underline{\mathrm{Morse}}}$ be space of generalized Morse fn on W. (all fn have ∂W as regular level set)

We have the diagram

$$\begin{tikzcd}
\underline{\underline{\mathrm{Stein}}} \arrow[rr, "\underline{\underline{W}}"] \arrow[dr, "\pi_{\underline{\underline{S}}}"'] & & \underline{\underline{\mathrm{Weinstein}}} \arrow[dl, "\pi_{\underline{\underline{W}}}"] \\
& \underline{\underline{\mathrm{Morse}}} &
\end{tikzcd}$$

where $\pi_{\underline{\underline{S}}}(J,\phi) = \pi_{\underline{\underline{W}}}(\omega, X, \phi) = \phi$.

Consider fibers

$$\underline{\underline{\mathrm{Stein}}}(\phi) := \pi_{\underline{\underline{S}}}^{-1}(\phi)$$

$$\underline{\underline{\mathrm{Weinstein}}}(\phi) := \pi_{\underline{\underline{W}}}^{-1}(\phi).$$

Last week, we learned that

$$\underline{\underline{\mathfrak{W}}}_\phi := \underline{\underline{\mathfrak{W}}}|_{\underline{\underline{\mathrm{Stein}}}(\phi)} : \underline{\underline{\mathrm{Stein}}}(\phi) \to \underline{\underline{\mathrm{Weinstein}}}(\boxed{\phi})$$

$\boxed{\text{fixed!}}$

is a weak-homotopy equivalence.

But what if ϕ is allowed to vary?
Then what can we say?

Today, we prove 2 main thms.

<u>Thm 1.1(b)/15.3</u>: Given a Weinstein homotopy $\underline{\underline{\mathfrak{W}}}_t = (\omega_t, X_t, \phi_t)$, $t \in [0,1]$ on W starting at a Stein structure, $\underline{\underline{\mathfrak{W}}}_0 = \underline{\underline{\mathfrak{W}}}(J, \phi)$, there is a Stein homotopy (J_t, ϕ_t) starting at $(J_0, \phi_0) = (J, \phi)$ s.t. $\underline{\underline{\mathfrak{W}}}(J_t, \phi_t)$ and $\underline{\underline{\mathfrak{W}}}_t$ are homotopic w/ fixed fn. ϕ_t and fixed at $t=0$.

Moreover, $\exists$ diffeotopy $h_t : W \to W$ with $h_0 = \mathrm{Id}$ s.t. $h_t^* J_t = J \quad \forall t \in [0,1]$.

Thm 1.1(c) / Thm 15.4 (but I think 15.4 is written wrong) : Given a Weinstein homotopy $\underline{\underline{W}}_t = (\omega_t, X_t, \phi_t)$, $t \in [0,1]$ on W connecting $\underline{\underline{W}}_0 = \underline{\underline{W}}(J_0, \phi_0)$ and $\underline{\underline{W}}_1 = \underline{\underline{W}}(J_1, \phi_1)$ with $\phi_t = \phi_1$ for $t \in [1/2, 1]$.

There is a Stein homotopy (J_t, ϕ_t) connecting (J_0, ϕ_0) and (J_1, ϕ_1) s.t. the paths $\underline{\underline{W}}(J_t, \phi_t)$ and $\underline{\underline{W}}_t$ are homotopic w/ fixed fn ϕ_t and fixed at 0, 1.

Now to reframe these, introduce the following spaces:

$$\mathcal{D}_J(\phi) := \{ h \in \mathcal{D} \mid \phi \text{ is } h^*J \text{ convex} \}$$

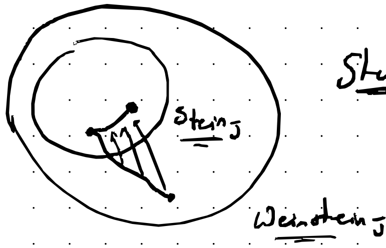
$\mathcal{D}$: diffeomorphisms diffeotopic to identity.

"homotopy fibers"

$$\mathcal{P}_J(\phi) := \{ (h, \gamma) \mid h \in \mathcal{D}_J(\phi),\ \gamma : [0,1] \to \underline{\text{Weinstein}}(\phi),\ \gamma(0) = \underline{\underline{W}}(h^*J, \phi) \}$$

$$\mathcal{P}_J := \bigcup_{\phi \in \underline{\text{Morse}}} \mathcal{P}_J(\phi).$$

Consider the map $\pi_{\mathcal{P}} : \mathcal{P}_J \to \underline{\underline{\mathrm{Weinstein}}}_J$, $(h,\gamma) \mapsto \gamma(1)$.



$$\underline{\underline{\mathrm{Stein}}}_J := \left\{ \underline{\underline{W}}(h^*J, \phi),\ h \in \mathcal{D}_J(\phi) \right\}$$

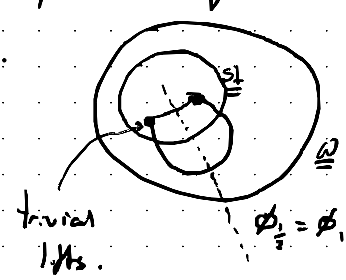
1.1(b) tells us that I satisfy path-lifting lifting for paths beginning in $\underline{\underline{\mathrm{Stein}}}_J$

1.1(a) tells us that if I define analogously

$$\mathcal{P}(\phi) := \left\{ (J,\gamma) \,\middle|\, (J,\phi) \in \underline{\underline{\mathrm{Stein}}},\ \gamma : [0,1] \to \underline{\underline{\mathrm{Weinstein}}}(\phi) \atop \gamma(0) = \underline{\underline{W}}(J,\phi) \right\}$$

$$\mathcal{P} := \bigcup_{\phi \in \underline{\mathrm{Morse}}} \mathcal{P}(\phi),$$

that then $\pi_{\mathcal{P}} : \mathcal{P} \to \underline{\underline{\mathrm{Weinstein}}}$, $(h,\gamma) \mapsto \gamma(1)$ also has lift extension for specific classes of paths beginning and ending at $\underline{\underline{\mathrm{Stein}}}$.



Note that right now all we have are based lifting from Stein basepts and restricted extensions.

A conjecture is that we can upgrade 1.1(b) so that

$$\mathrm{Thp}_J : \mathcal{P}_J \to \underline{\mathrm{Weinstein}}_J$$

is a Serre fibration.

Note that if this is the case, then Corollary A.6 tells us that since $\underline{\mathrm{Weinstein}}_J$ is path-connected, and $\mathcal{P}(\emptyset) \to \underline{\mathrm{Weinstein}}(\emptyset)$ is a weak homotopy equivalence, that then we have that

$$\underline{W} : \underline{\mathrm{Stein}} \to \underline{\mathrm{Weinstein}}$$

is a weak homotopy equivalence as well.

This is the hope.

Our results provide an initial blueprint for this result.

§2 Proving the main thing

$\boxed{\text{Prelude to 15.3}}$

- Without going into too much detail, recall that a Weinstein homotopy on a Weinstein manifold is a concatenation of simple Weinstein homotopies, i.e., $\underline{\underline{W}}_t = (W, \omega_t, X_t, \phi_t)$ is s.t. $\exists$ a partition $0 = t_0 < \cdots < t_N = 1$ and unbdd sequences of smooth fn. $c_1^k(t) < c_2^k(t) < \cdots$, $t \in [t_{k-1}, t_k]$, $k = 1, \ldots, N$ s.t. $c_j^k(t)$ is a regular value for ϕ_t $\forall t \in [t_{k-1}, t_k]$.

One can reparametrize ϕ_t to assume $c_i^k(t)$ is constant. (see Remark 11.25)

Now we can work by exhaustion and consider on each $[t_{k-1}, t_k]$ the cobordisms

$$W_j^k(t) := \{ c_j^k \leq \phi_t \leq c_{j+1}^k \}$$

One can produce a diffeotopy f_t (perhaps by considering the flow of some time dependent vector field) that fixes

$$\Sigma_j^k(t) := \partial^- W_j^k(t) = \{ c_j^k = \phi_t \},$$

for all t after replacing $\underline{\underline{W}}_t$ with $(f_t)_* \underline{\underline{W}}_t$.

Hence, if we can prove the cobordism version, with diffeotopies and homotopies fixed near the boundaries up to scaling, then we can glue these Stein homotopies along the exhausting cobordisms to yield the general result.

In other words, it suffices to prove:

Thm 15.1 (First Stein deformation thm).

If $\underline{\underline{W}}_t = (W, \omega_t, X_t, \phi_t)$ is a homotopy of Weinstein cobordisms w/ $\underline{\underline{W}}_0 = \underline{\underline{W}}(J, \phi_0)$ for a Stein str. (J, ϕ_0) on W, then after target repar. of ϕ_t, $\exists$ diffeotopy $h_t : W \to W$ Op ∂W with $h_0 = \mathrm{Id}$ s.t. $h_{t*}\phi_t$ are J-convex and with $\underline{\underline{W}}_t$ and $\underline{\underline{W}}(h_t^* J, \phi_t)$ homotopic w/ fixed fn. ϕ_t and fixed at $t=0$.

If $\underline{\underline{W}}_t$ is fixed near ∂W and/or fixed up to scaling near $\partial_+ W$, the same can be arranged for the homotopy of paths.

Proving the cobordism version of our thm will be the main goal for our talk.

Notice we can modify our original topological pictures by adding extra boundary conditions (i.e., Stein near $\partial_- W$, paths fixed near $\partial_- W$, etc...)

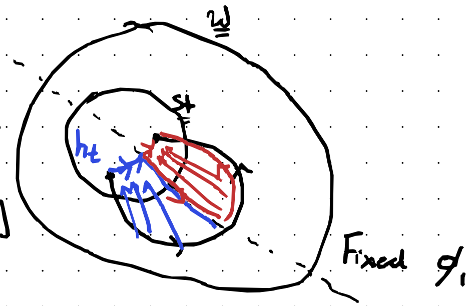
for the cobordism case. (see pg. 297).

Now before we proceed, let's show how Thm 1.1(c) follows immediately from Thm 15.3 and the weak homotopy equivalence of

$\underline{\mathrm{Stein}}(\phi) \to \underline{\mathrm{Weinstein}}(\phi)$. (WHE)

Indeed, notice the 1^{st} half $t \in [0, \frac{1}{2}]$ can be framed in context of Thm 15.3 to yield a diffeotopy h_t inducing a Stein homotopy that is path homotopic to the 1^{st} half of our path, call it $\mathfrak{W}(J_t, \phi_t)$. $t \in [0, \frac{1}{2}]$

In particular, we get a path from



$\underline{\underline{W}}(J_{1/2}, \phi_{1/2})$ to $\underline{\underline{W}}_{1/2}$, which we can concatenate with $\underline{\underline{W}}_t$ to get a path in $\underline{\underline{\text{Weinstein}}}(\phi_1)$. WHE then lifts this to a path $\underline{\underline{W}}(J_{1/2}, \phi_{1/2}) \longrightarrow \underline{\underline{W}}(J_1, \phi_1)$ in $\underline{\underline{\text{Stein}}}(\phi_1)$.

The cobordism analogue Thm 15.2 has the same pf. when derived from Thm 15.1, with only changes being some extra conditions on the bdry components.

statement &

So what's left is to prove Thm 15.1.

The key idea is to reduce to the elementary homotopy case via an "admissible partition" (Lemma 9.37).

Let $\underline{\underline{W}}_t = (W, \omega_t, X_t, \phi_t)$ be a homotopy of Weinstein cobordisms. Essentially, an admissible partition yields a "time" partition

$$0 = t_0 < \cdots < t_p = 1$$

and a "space" partition in the form of a finite sequence of fn.

$$m(t) = c_0^k(t) < c_1^k(t) < \cdots < c_{N_k}^k(t) = M(t)$$

for each $k = 1, \ldots, p$, $t \in [t_{k-1}, t_k]$ where

$$m(t) := \phi_t(\partial W) \quad \text{and} \quad M(t) := \phi_t(\partial_+ W)$$

s.t. $c_j^k(t)$, $j = 0, \ldots, N_k$ are regular values of ϕ_t and each Cerf–Morse homotopy

$$\underline{W}_t \big|_{W_j^k(t)}$$

$$W_j^k(t) := \{ c_{j-1}^k(t) \leq \phi_t \leq c_j^k(t) \}$$

is elementary.

In the same way as for the manifold → cobordism reduction, arrange so that $c_j^k = c_j^k(t)$ and

$$\Sigma_j^k = \{ \phi_t = c_j^k(t) \}$$ are independent of

$t \in [t_{k-1}, t_k]$ via some "twisting" diffeotopy.

Now we want to proceed by induction.

Note $\partial_- W_j^k = \Sigma_{j-1}^k$, $\partial_+ W_j^k = \Sigma_j^k$.

Recall that what we want is a diffeotopy $h_t : W \to W$ rel $Op\,\partial W$ with $h_0 = id$ s.t. $h_{t*}\phi_t$ are J-convex and a homotopy of paths $\underline{\underline{W}}^s_t$ s.t.

- $\underline{\underline{W}}^0_t = \underline{\underline{W}}_t$, $\underline{\underline{W}}^1_t = \underline{\underline{W}}(h_t^* J, \phi_t)$
- $\underline{\underline{W}}^s_t$ has fixed ϕ_t
- $\underline{\underline{W}}^s_0 = \underline{\underline{W}}_0 = \underline{\underline{W}}(J, \phi_0)$ for all s
- $\underline{\underline{W}}^s_t$ fixed near ∂W and/or fixed up to scaling near $\partial_+ W$ if $\underline{\underline{W}}_t$ is.

Now sps $\underline{\underline{W}}^s_t$ and h_t as above exist for $t \leq t_{k-1}$. How do we extend it to the interval $[t_{k-1}, t_k]$? Now if we want to patch things, we must establish a good "patching environment." In particular, we want

$$\underline{\underline{W}}_t \big|_{Op\, \Sigma_j^k} = \underline{\underline{W}}\left(h^*_{t_{k-1}} J, \phi_{t_{k-1}}\right)\Big|_{Op\, \Sigma_j^k}.$$

So things are fixed near the ∂W_j^k.

This way, if we know how to extend things in the elementary case, with our homotopy of paths fixed on the bdry, we can patch these together with a simple reparametrization.

So let's try to perform such a reduction.

Observe that since we "froze" Σ_j^k, each Σ_j^k has a nbhd without crit pts of ϕ_t. This nbhd is up to isotopy a product collar

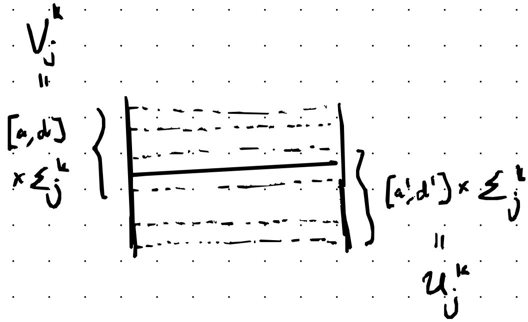
$$\mathcal{U}_j^k \cong [a,d] \times \Sigma_j^k \qquad \begin{pmatrix} \text{say,} \\ a = c_j - \varepsilon \\ d = c_j + \varepsilon \end{pmatrix}$$

Now Lemma 12.7 tells us that if I look at a subcobordism of this nbhd, say.

$$(\mathcal{U}_j^k)' = ([a,b] \times [c,d]) \times \Sigma_j^k, \quad a < b < c < d$$

then restriction of Weinstein structures from $\mathcal{U}_j^k$ to $(\mathcal{U}_j^k)'$ is a Serre fibration.

In particular, I can look at 2 such restrictions around Σ_j^k as in figure:



Now if on

- $[a,b] \times \Sigma_j^k$ I made things constant
  $[c',d'] \times \Sigma_j^k$ $= \underline{\underline{\mathcal{W}}}(h_{t_{k-1}}^* J, \phi_{t_{k-1}})$
- $[c,d] \times \Sigma_j^k$, I make it agree w/
  $[a',b'] \times \Sigma_j^k$ $\mathcal{W}_t$

I can prescribe a Weinstein homotopy on $\mathcal{U}_j^k \cup V_j^k$ that is constant $\underset{\underline{\underline{\mathcal{W}}}^1_{t_{k-1}}}{=} \underline{\underline{\mathcal{W}}}(h_{t_{k-1}}^* J, \phi_{t_{k-1}})$ on a small nbhd of Σ_j^k but $= \underline{\underline{\mathcal{W}}}_t$ on the bdry of $\mathcal{U}_j^k \cup V_j^k$, so it doesn't affect the rest of the homotopy. Doing this near each Σ_j^k $\forall j$ gives us what we need. Let's summarize our reductions:

Break up into elementary pieces
"admissible partition"

$\downarrow$ (diffeotopy)

Arrange the partition to have "frozen" pieces for the boundaries, stabilizing the cobordisms we want to induct over

$\downarrow$ Lemma 12.7

Given the induction hypothesis / $\underline{\mathcal{W}}^s_t$ for $t \leq t_{k-1}$, modify $\underline{\mathcal{W}}_t$ so it agrees with our $\underline{\mathcal{W}}^1_{t_{k-1}} = \underline{\mathcal{W}}(h^+_{t_{k-1}} J, \phi_{t_{k-1}})$ near the boundaries of the elementary pieces.

"Make patching possible"

So now we simply need to prove the following, which will be our induction step. One can guess what we need from what we have from our induction hypothesis.

Lemma 15.10

- Let (W, J, ϕ) be a Stein cobordism and $\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t)$, $t \in [0,1]$ be an elementary Weinstein (WS) homotopy s.t. $\mathfrak{W}_t = \mathfrak{W}(J, \phi)$ on $\mathcal{O}p(\partial W)$.

Spse we are given a W.S. homotopy $\mathfrak{W}_0^s$, $s \in [0,1]$ with fixed fn. ϕ and fixed on $\mathcal{O}p\, \partial W$ from $\mathfrak{W}_0^0 = \mathfrak{W}_0$ to $\mathfrak{W}_0^1 = \mathfrak{W}(J, \phi)$

(exactly our induction hypothesis!)

Then after target reparam. ϕ_t, $\exists$ diffeotopy $h_t : W \to W$ fixed on $\mathcal{O}p\, \partial W$ with $h_0 = Id$ s.t. $h_{t*}\phi_t$ are J-convex.

Moreover, the Weinstein homotopy $\mathfrak{W}_0^s$ extends to a homotopy of paths of WS structures $\mathfrak{W}_t^s$ $s, t \in [0,1]$ <u>fixed on $\mathcal{O}p\, \partial_- W$</u> and <u>up to scaling on $\mathcal{O}p\, \partial_+ W$</u>, with fixed fn ϕ_t from $\mathfrak{W}_t^0 = \mathfrak{W}_t$ to $\mathfrak{W}_t^1 = \mathfrak{W}(h_t^* J, \phi_t)$.

Before we proceed, let's show how this result finishes our proof.

Indeed, after our reductions we obtain a homotopy, which we'll still denote as $\underline{\underline{\omega}}_t$, $t \in [t_{k-1}, t_k]$ that agrees on $\mathcal{O}p\, \Sigma_j^k$ for all j and $t \in [t_{k-1}, t_k]$, still elementary over each cobordism W_j^k.

Applying Lemma 15.10 to this yields a diffeotopy

$$h_t^j : W_j^k \to W_j^k, \quad t \in [t_{k-1}, t_k] \text{ that is}$$

Fixed on $\mathcal{O}p\, \partial W_j^k$ with $h_0^j = \mathrm{id}$ and $h_{t_*}^j$ Jzmex.

along with Weinstein homotopies of paths

$\underline{\underline{\omega}}_t^s|_{W_j^k}$ extending $\underline{\underline{\omega}}_{t_{k-1}}^s$ s.t.

- $\underline{\underline{\omega}}_t^0|_{W_j^k} = \underline{\underline{\omega}}_t|_{W_j^k}$ and $\underline{\underline{\omega}}_t^1|_{W_j^k} = \underline{\underline{\omega}}(h_t^* \mathfrak{J} \phi_t)|_{W_j^k}$

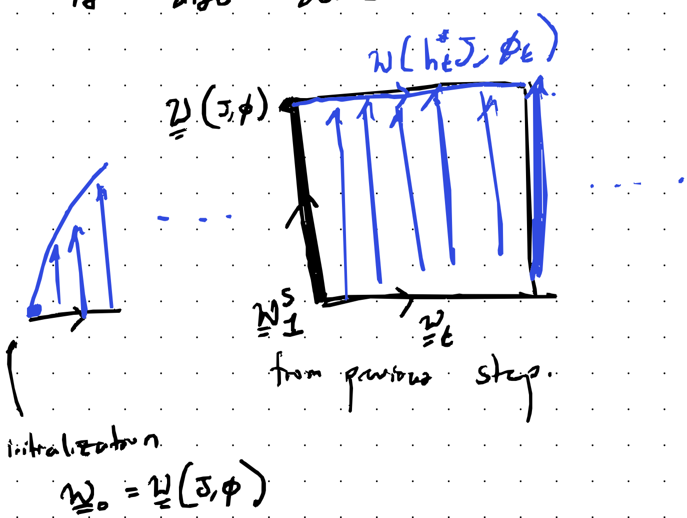
- $\underline{\underline{\omega}}_t^s|_{W_j^k}$ is fixed on $\mathcal{O}p\, \partial W_j^k$.

Since $h_t^j = \mathrm{id}$ on $\mathcal{O}p\, \partial W_j^k$, these patch to a global diffeotopy $\tilde{h}_t : W \to W$, $t \in [t_{k-1}, t_k]$.

Agreement up to scaling on $O_p(\partial W_j^k)$ also allow us to patch to a global extension $\underline{W}_t^s$ over $[t_{k-1}, t_k]$ fixed on $O_p \partial W_-$ and up to scaling on $O_p \partial W_+$.

Note for $t=0$, the assumptions of Lemma 15.10 trivially apply, so our base case is also done.

$\boxtimes$ Thm 15.1



So what is the strategy for L15.10?

The idea is to recall that up to diffeotopy, any two WS homotopies with the same profile can be deformed into each other (Lemma 12.23).

So we want to replace our given family f of f_n by a J-convex model family $\tilde{\phi}_t$

ϕ_t

with the same profile, then transfer the WS homotopy from one to the other via the diffeotopy produced by L12.23.

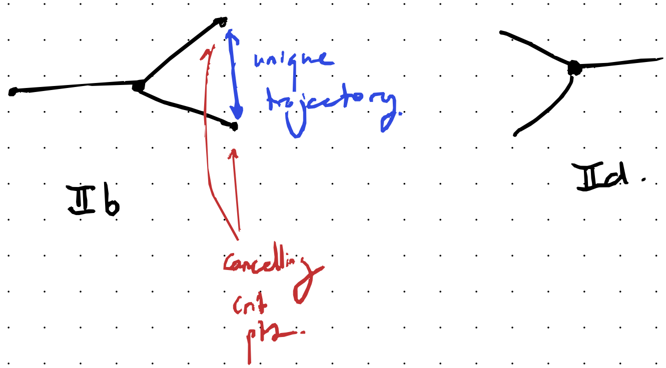
Consider the following special case:

Spse $\underline{\underline{W}}_0 = \underline{\underline{W}}(J, \phi)$.

Now recall we have three types of elementary Morse homotopies. $\rightarrow$ I $\rightarrow$ No bifurcation

IIb $\rightarrow$ Birth

IId $\rightarrow$ Death.



Note Thm 10.10 – 10.12 give us thms that allow us to produce J-convex families of fn. ϕ_t up to target rep. for each of these types starting at ϕ_0.

By reparametrizing the target par. and the homotopy parameter, one can arrange that the J-convex families of fns $\tilde{\phi}_t$ have the same profile as ϕ_t.

L12.23 then gives a diffeotopy h_t s.t.

$$\phi_t = \tilde{\phi}_t \circ h_t$$

and a homotopy $\underline{\underline{\mathfrak{W}}}_t^s$ s.t.

$$\underline{\underline{\mathfrak{W}}}_t^0 = \underline{\underline{\mathfrak{W}}}_t \quad \text{and} \quad \underline{\underline{\mathfrak{W}}}_t^1 = h_t^* \underline{\underline{\mathfrak{W}}}(J, \tilde{\phi}_t)$$

$$= \mathfrak{W}(h_t^* J, \underbrace{\tilde{\phi}_t \circ h_t}_{= \phi_t}).$$

in particular, $h_{t*}\phi_t$ is J-convex.

All other properties (bdry) are satisfied because

L10.10-10.12 and L12.23 all respect stabilization near bdry.

Great. So all that's left is to show that we can always reduce to the case $\underline{\underline{\mathfrak{W}}}_0 = \underline{\underline{\mathfrak{W}}}(J, \phi_0)$.

What do we know about our initial conditions?
We have a path $\underline{\underline{\mathfrak{W}}}_0^s$ from $\underline{\underline{\mathfrak{W}}}_0$ to $\underline{\underline{\mathfrak{W}}}_0^1 = \underline{\underline{\mathfrak{W}}}(J, \phi)$ s.t. $\underline{\underline{\mathfrak{W}}}_0^s$ has fixed fn. $\phi_0 = \phi$ and $\underline{\underline{\mathfrak{W}}}_0^s$ is fixed on $\mathcal{O}_p \partial W$.

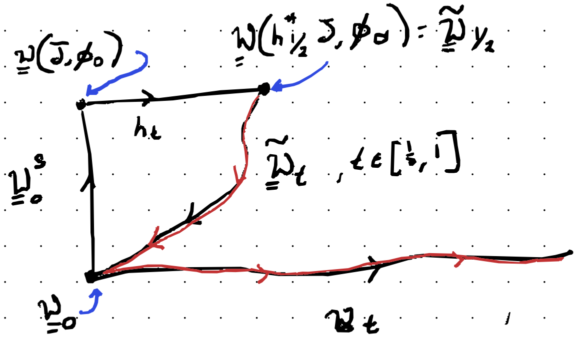
The reason we could apply L10.10-10.12 was because the path, beginning at $\underline{\underline{\mathfrak{W}}}_0 = \underline{\underline{\mathfrak{W}}}(J, \phi)$ was known to be elementary of some type. But the path $\underline{\underline{\mathfrak{W}}}_0^s$ need <u>not</u> be elementary of that same type!

To remedy this, we show that we can create a homotopy $\tilde{\underline{\underline{\mathfrak{W}}}}_t$ s.t.

$\tilde{\underline{\underline{\mathfrak{W}}}}_0 = (W, \tilde{\omega}_t, \tilde{X}_t, \tilde{\phi}_t)$ is Stein on $t \in [0, \frac{1}{2}]$, in the sense that it is induced on the 1^{st} half by a diffeotopy $h_t : W \to W$ $t \in [0, \frac{1}{2}]$ with ϕ_0 $h_t^* J$-convex. i.e., $\tilde{\underline{\mathfrak{W}}}_t = \underline{\mathfrak{W}}(h_t^* J, \phi_0)$ and on the 2^{nd} half, it $\boxed{\text{concatenated to something}}$ elementary of the type we need, either Type I or IId.

Note for IIb this reduction is not necessary since at $t=0$, $\phi_0 = \phi$ has no critical pts. Hence, our initial homotopy $\underline{\mathfrak{W}}_0^s$ trivially concatenated with $\underline{\mathfrak{W}}_t$ to produce a birth type homotopy since $\underline{\mathfrak{W}}_0^s$ has fixed fn. ϕ_0 in this case. So no reduction is needed to apply the Creation thm to get a family of the same profile.

To see how this reduction works, consider the following diagram



- red path elementary.

Notice the red path satisfies our assumptions for Lemma 15.10.

So essentially, we modify the initial boundary value problem to one that begins with a trivial Stein lift.

Now you might wonder how we recover an extension to $\underline{\underline{\mathcal{W}}}_0^s$. Indeed, it seems that the lift of the red path does not patch with $\underline{\underline{\mathcal{W}}}_0^s$

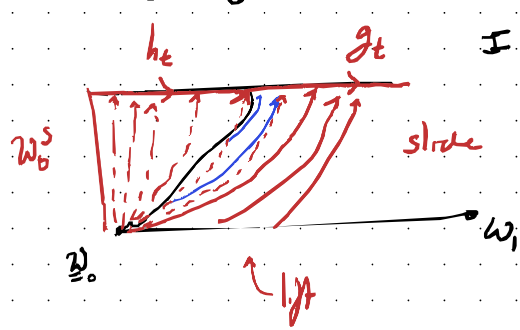
Furthermore, the lift of $\underline{\underline{\mathcal{W}}}_t$ does not seem to be aligned with the lift of $\underline{\underline{\mathcal{W}}}_0$, since we also apply the lift in our reduced problem to $\tilde{\mathcal{W}}_t$, $t \in [\frac{1}{2}, 1]$.

Question: How can we use this diffeotopy to produce an "aligned" extension? How does answering the reduced problem answer our original problem?

In fact, more generally

Whenever we perform such diffeotopies and modifications, how do we recover a solution to our original problem which demands "alignment"?

My guess: You undo the diffeotopy and slide back lift via the diffeotopy after you get it (do you pull back the whole path? I couldn't figure out details)



slide back along $h_t^{-1} \circ g_t^{-1}$

$\square$

Assuming the reduction is valid, let us proceed. Recall we have two types of normalization we need to consider. Type I and Type IId.

Let's begin with the lemma for the Type I procedure.

Lemma 15.8: Let (ω, J, ϕ) be a Stein cobordism and $\underline{\underline{W}} = (W, \omega, X, \phi)$ an elementary WS cobordism s.t. $\underline{\underline{W}} = \underline{\underline{W}}(J, \phi)$ on $\mathcal{O}_p \partial W$. Spse that $\underline{\underline{W}}$ and $\underline{\underline{W}}(J, \phi)$ are connected by a WS homtpy w/ fixed fn. ϕ and fixed on $\mathcal{O}_p \partial W$.

Then after reparam. ϕ, $\exists$ WS hmtpy $\underline{\underline{W}}_t = (W, \omega_t, X_t, \phi_t)$, $t \in [0,1]$ s.t.

- $\underline{\underline{W}}_0 = \underline{\underline{W}}(J, \phi)$, $\underline{\underline{W}}_1 = \underline{\underline{W}}$.
- $\underline{\underline{W}}_t$ fixed on $\mathcal{O}_p \partial_- W$, up to scaling on $\mathcal{O}_p \partial_+ W$

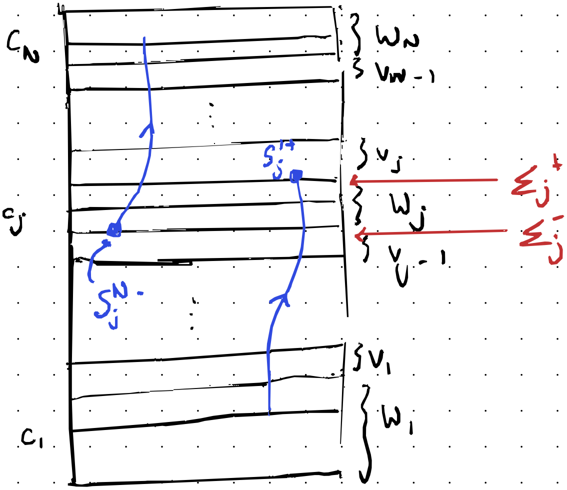
(Stein on 1st half)
- For $t \in [0, 1/2]$, ϕ is $h_t^* J$-convex, $\underline{\underline{W}}_t = \underline{\underline{W}}(h_t^* J, \phi)$ for a diffeotopy $h_t : W \to W$, $t \in [0, 1/2]$, $h_0 = Id$, $h_t|_{\mathcal{O}_p \partial W} = Id$.

(Elementary on 2nd half)
- For $t \in [1/2, 1]$, $\underline{\underline{W}}_t$ elementary with attaching spheres in $\partial_- W$ of all crit pts of ϕ fixed

Our strategy for this will be similar for the Pf of the Thm. 15.1 We break up W into pieces on which the homotopy is easier to define, reduce our problem so that it respects this partition and allows for gluing of solutions between pieces, and then patch things together to get our desired $\underline{\underline{\omega}}_t$. The key data we need to track are that of the attaching spheres, since we want our final homotopy to be elementary on the interval $[\frac{1}{2}, 1]$ with the intersection of the stable mflds with $\partial_- W$ unchanged.

So we break up W as such:

Let $c_1 < \cdots < c_N$ be crit values of ϕ. Set $c_0 = \phi|_{\partial_- W}$ and $c_{N+1} := \phi|_{\partial_+ W}$. Choose ε small and partition W into subcobordisms.



$$W_j := \{c_j - \varepsilon \le \phi \le c_j + \varepsilon\}, \quad j = 2, \dots, N-1$$

$$W_1 := \{\phi \le c_1 + \varepsilon\}, \quad W_N := \{\phi \ge c_N - \varepsilon\}.$$

$$V_j := \{c_j + \varepsilon \le \phi \le c_{j+1} - \varepsilon\} \quad j = 1, \dots, N-1$$

$$\Sigma_j^{\pm} := \{\phi = c_j \pm \varepsilon\} \quad j = 1, \dots, N.$$

Observe $\Sigma_j^+ = \partial_- V_j = \partial_+ W_j$ for $j = 1, \dots, N-1$

$\Sigma_j^- = \partial_+ V_j = \partial_- W_j$ for $j = 2, \dots, N$.

Write $\xi_j^{\pm}$ for the contact structure form on induced by the Liouville form on $\Sigma_j^{\pm}$.

For $k \ge j$ denote by S_j^{k-} the intersection of the union of the X-stable mfds of the crit pts on level c_k

with the hypersurface Σ_j^-,

Similarly for $i \leq j$, denote by S_j^{i+} the intersection of the union of the X-unstable mfds of the crit pts on level c_i with the hypersurface Σ_j^+. Set

$$\mathcal{S}_j^- = \bigcup_{k \geq j} S_j^{k-}, \quad \mathcal{S}_j^+ = \bigcup_{i \leq j} S_j^{i+}$$

We keep track of these intersections because we want to make sure $\underline{W}_t$ does not cause these intersections to "slide" across and possibly create new trajectories.

To summarize what we have done - we have broken W down so that each piece contains at most one critical value (like writing W as sequence of handle attachments) and are simultaneously keeping track of attachment data on the boundaries of these critical pieces to keep our homotopy elementary.

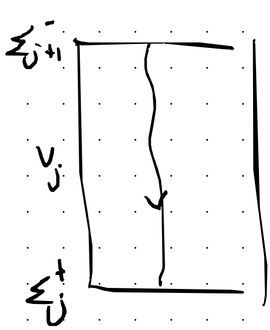
Spse we have the special case where $\underline{\underline{\mathfrak{W}}} = \underline{\underline{\mathfrak{W}}}(J,\phi)$ on $Op\left(\bigcup_{j=1}^{N} W_j\right)$ and $\bar{\underline{\underline{\mathfrak{W}}}}_t$ a homotopy between them fixed on $Op(\partial W \cup \mathrm{Crit}\,\phi)$ such that $\bar{\underline{\underline{\mathfrak{W}}}}_t$ induces the same contact structure on all level sets.

Then we can construct our homotopy separately on each V_j, fixed near $\partial_- V_j$ and up to scaling near $\partial_+ V_j$.

This will allow us to extend our homotopy to $\bigcup_{j=1}^{N} W_j$ as fixed up to scaling.

Write $\bar{\underline{\underline{\mathfrak{W}}}}_t = (V_j, \bar{\omega}_t, \bar{X}_t, \phi)$ $t \in [0,1]$. $\left(\text{for } \bar{\underline{\underline{\mathfrak{W}}}}_t|_{V_j}\right)$

Recall holonomy along $\bar{X}_t$ defines contactomorphisms



$$T_{X_t} : (\Sigma_{j+1}^-, \xi_{j+1}^-) \to (\Sigma_j^+, \xi_j^+)$$

Note ϕ has no crit values on V_j and is J-convex with

$$T_{\bar{\underline{\underline{\mathfrak{W}}}}_{1-t}}(\Lambda_{j+1}^-) := L_t$$

an isotropic isotopy with

$$L_0 := \Pi_{\overline{\omega}_1}(\mathbb{S}^-_{j+1}) = \text{"image of } \mathbb{S}^-_{j+1} \text{ under negative grad flow of } -\nabla_\phi \phi\text{"}$$

$$\overline{\omega}_1 = \underline{\underline{W}}(J,\phi)|_{V_j}$$

beginning at the image of the negative J-gradient flow. [after repar. ϕ_j]

Then Prop 10.1 tells us that we can find a diffeotopy $h_t : V_j \to V_j$ that has

- $h_t = Id$ on $\mathcal{O}p\, \partial V_j$
- $\phi_t := \phi \circ h_t$ J-convex
- $L_t = \Pi_{X_t^1}(\mathbb{S}^-_{j+1})$

where $X_t^1 := -\nabla_{\phi_t} \phi_t$.

So we get a homotopy

$$\underline{\underline{W}}_t^1 = (V_j, \omega_t^1, X_t^1, \phi) := \underline{\underline{W}}(V_j, h_t^* J, \phi)$$

s.t.

$$\Pi_{X_t^1}(\mathbb{S}^-_{j+1}) = \Pi_{\overline{X}_t}(\mathbb{S}^-_{j+1})$$

$\lceil$ Detail I haven't fleshed out:

Does $\underline{\underline{W}}_t^1$ induces same contact structures on level sets?

If not, how to arrange it to be so? $\rfloor$

Assuming it is so, or can be made to be so via some Gray type argument that $\overline{\underline{\underline{W}}}_t$ and $\underline{\underline{W}}_t'$ induce the same contact structures on all level sets, note that Lemma 12.6 then yields the type of path we want, i.e., a path

$$\hat{\underline{\underline{W}}}_t = (V_j, \hat{\omega}_t, \hat{X}_t, \phi)$$

s.t. • $\hat{\underline{\underline{W}}}_{2t} = \underline{\underline{W}}'_{2t}$ for $t \in [0, \frac{1}{2}]$

Stein!, obtained via diffeotopy

• $\hat{\underline{\underline{W}}}_1 = \overline{\underline{\underline{W}}}_0 = \underline{\underline{W}}$.

in the lemma, we need concerned hmtpys to start at same point.

• $\hat{\underline{\underline{W}}}_t$ induces same contact str. on level sets.

• $T_{\hat{X}_t}(\mathcal{S}^-_{j+1}) = T_{X_1'}(\mathcal{S}'_{j+1})$ for $t \in [\frac{1}{2}, 1]$

$\underline{\underline{W}}_{1-} = \underline{\underline{W}}_0$

things stay elementary on 2^{nd} half.

Note that this resulting homotopy gives us everything we want.

So the question is how to get these reductions.

Why can we assume $\underline{\omega} = \underline{\underline{W}}(J, \phi)$ on $\mathcal{O}_p\left(\bigcup_{j=1}^{N} W_j\right)$ and $\underline{\underline{\overline{W}}}_t$ fixed on $\mathcal{O}_p(\partial W_0 \cup \mathrm{Crit}\,\phi)$?

First, observe that there are charts locally around crit. pts in which stable and unstable mfds of (X_J, ϕ) and $(\overline{X}_{1-t}, \phi)$ are given locally by coord planes around each crit. pt p, where ϕ has the same form.

This allows us to construct local diffeomorphisms
$h_{p,t} : \mathcal{O}_p\, p \to \mathcal{O}_p\, p$ where $\phi \circ h_{t,p} = \phi$
and where $(\overline{X}_{1-t}, \phi)$ and $\left(h_{t,p}^{*}(X_J), \underbrace{\phi \circ h_{tp}}_{\phi}\right)$ have the same stable & unstable mfds.

Extend this to a global diffeotopy h_t

$\boxed{\text{Question: How to do this in detail?}}$

s.t. $h_0 = \mathrm{Id}$, $h_t|_{\mathcal{O}_p \partial W} = \mathrm{id}$, $\phi \circ h_t = \phi$.

Note that now, though stable and unstable mflds agree on $\mathcal{O}_p(\partial W \cup \mathrm{Crit}\,\phi)$, the Weinstein structures still need to be modified.

Prop 12.12, however, tells us that one can always modify one WS structure locally to another, around a crit pt as long as the crit pt has same value and index.

This modification can be made to preserve stable/unstable mflds if they agree.

Hence, $\exists$ WS homotopy $\tilde{\underline{\underline{W}}}_t = (W, \omega_t, X_t, \phi)$ rel $\mathcal{O}_p \partial W$ s.t. $\tilde{\underline{\underline{W}}}_0 = \underline{\underline{W}}$ and $\tilde{\underline{\underline{W}}}_1 = h_1^* \underline{\underline{W}}(J, \phi)$ near crit pts with stable and unstable mflds of all $\tilde{\underline{\underline{W}}}_t$ agreeing with $\underline{\underline{W}}$. In particular, $\tilde{\underline{\underline{W}}}_t$ stays elementary of Type I.

fixed

- Replace $\underline{\underline{W}}$ by $\tilde{\underline{\underline{W}}}_1$, J by $h_1^* J$ and assume $\underline{\underline{W}}$ agreed with $\underline{\underline{W}}(J, \phi)$ on $\mathcal{O}_p(\partial W \cup \mathrm{Crit}\,\phi)$.

Next, we want to assume that $\overline{\underline{\underline{\omega}}}_t$ is itself fixed on $\mathcal{O}_p(\partial W \cup \mathrm{Crit}\,\phi)$

But note that if we write

$$\underline{\underline{\omega}}_{loc} = \text{WS str. of } \underline{\underline{\omega}} \text{ or } \underline{\underline{\omega}}(J, \phi) \text{ on nbhd of } p$$

that we can apply 12.12 parametrically to get $\tilde{\underline{\underline{\omega}}}_t^s$ s.t. $\tilde{\underline{\underline{\omega}}}_t^0 = \overline{\underline{\underline{\omega}}}_t$

and $\tilde{\underline{\underline{\omega}}}_t^1 = \underline{\underline{\omega}} = \underline{\underline{\omega}}(J, \phi)$ on $\mathcal{O}_p(\partial W \cup \mathrm{Crit}\,\phi)$.

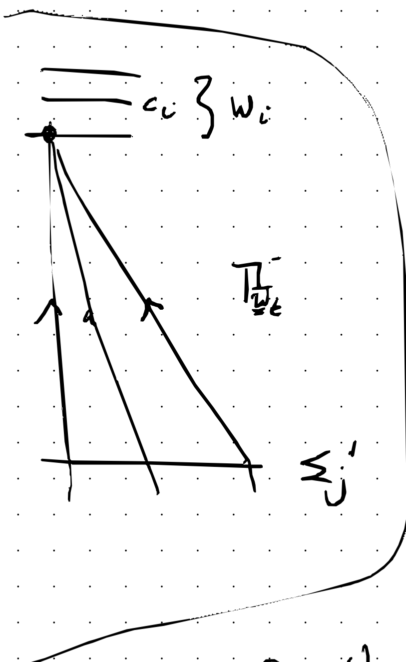
So we can assume $\overline{\underline{\underline{\omega}}}_t$ is fixed on $\mathcal{O}_p(\partial W \cup \mathrm{Crit}\,\phi)$.

Now $\overline{\underline{\underline{\omega}}}_t$ gives induced contact str. on $\phi^{-1}(c)$. As t varies, we get a contact isotopy. Gray tells us, then, $\exists\, g_t$ diffeo s.t. $g_t^* \overline{\underline{\underline{\omega}}}_t$ all have same contact str.

Relabel $g_t^* \underline{\underline{\omega}}_t \to \overline{\underline{\underline{\omega}}}_t$.

Since stable/unstable mflds of $\widetilde{\underline{W}}_t$ look the same locally, we can assume $\overline{X}_t$ unstable spheres on Σ_j^+ of crit pts on $\mathcal{G}$ are fixed

Now for $c_i > c_j$, I get intersections of stable mflds with Σ_j^+ that vary with $\widetilde{\underline{W}}_t$. These form an isotopy of isotropic spheres.



Isotopy extension yields a family of contactomorphisms $f_t : \Sigma_j^+ \to \Sigma_j^+$ that parametrize this isotopy.

But Lemma 12.5 tells us that these contactomorphisms can be realized as holonomy of a Weinstein homotopy.

$\widetilde{\underline{W}}_t = (\widetilde{\omega}_t, \widetilde{X}_t, \widetilde{\phi}_t)$ with intersections of $\widetilde{X}_t$ stable mflds of crit pts with $c_i > c_j$ with Σ_j^+ unchanging.

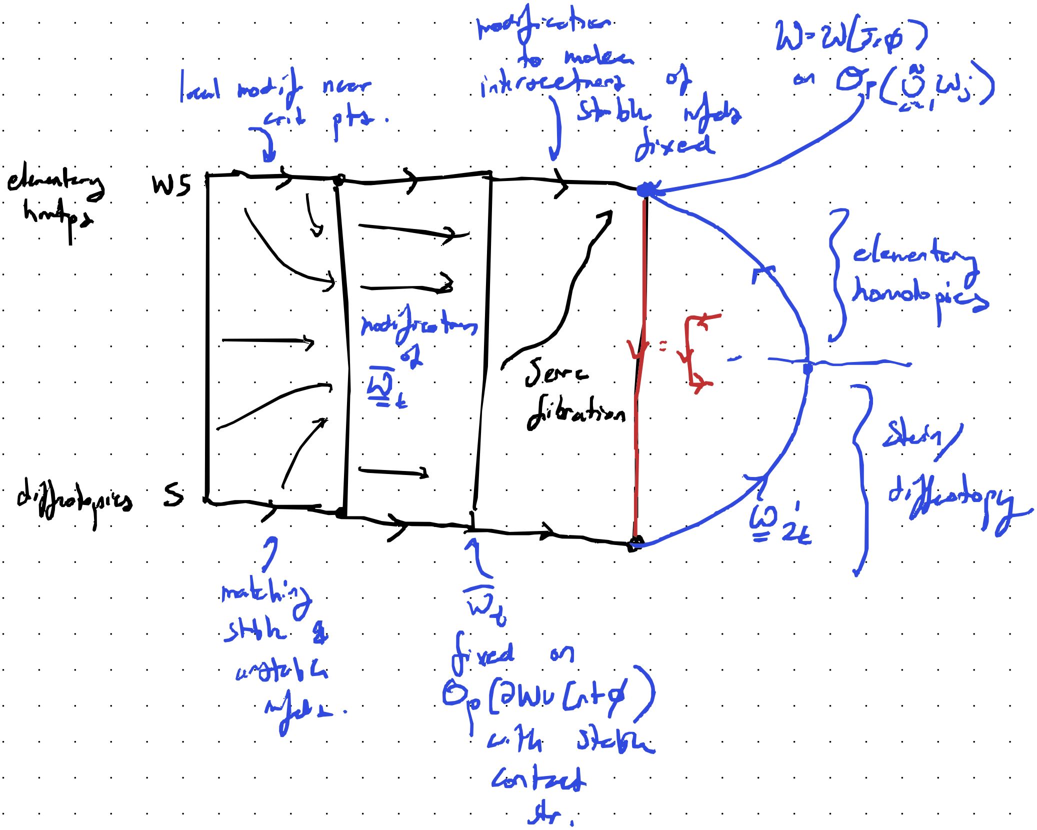
$\Rightarrow$ $\tilde{\underline{\underline{W}}}_t$ elementary $\forall t$

Hence, we can rename $\tilde{\underline{\underline{W}}}_1$ to $\underline{\underline{W}}$.

But note $\tilde{\underline{\underline{W}}}_1 = \underline{\underline{W}}_1$ on $\mathcal{O}p\left(\bigcup_{j=1}^{\infty} W_j\right)$!

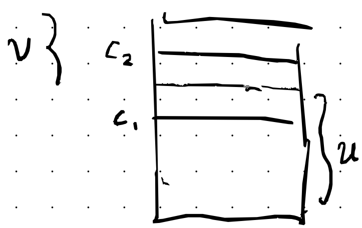
So we have the reduction we needed.

We summarize it here.



Recall that this gave us the final step in the reduction of the Birth type proof in the induction step.

Finally, for IId, we have pretty much the same Lemma, except for the "2nd part", we don't have something elementary, but something connected by a unique X_t-trajectory. We assume ϕ has 2 crit pts connected by a unique X-trajectory.
The broad sketch of this proof is to break W into two parts, one containing



the lower crit pt and one $\uparrow \mathcal{U}$ containing the higher V.

Reduce in the same way to assume $\underline{W} = \underline{W}(J, \phi)$ on $\mathcal{O}_p \mathcal{U}$. Then the restriction to V is elementary and we can apply 15.8. of type I

Since attaching sphere data is fixed, we get a unique trajectory.

This justifies our reduction in the induction step for Type IId.

We end the notes here: even though the author could write another 10 pages expanding on details, he realized (perhaps too late) that this is already quite excessive and he is also tired.

References:

- From Stein to Weinstein and Back: Symplectic Geometry of Affine Complex Manifolds

  — Kai Cieliebak, Yakov Eliashberg.