Synopsis

Weinstein manifolds are an important class of *open* symplectic manifolds that can be constructed in a topological fashion by assembling various standard pieces called "symplectic handles". Within symplectic geometry, they are important and arise naturally for several reasons:

- Many important results of differential topology based on Morse theory can be adapted into the symplectic world as results about Weinstein manifolds.
- One of the great open problems of symplectic topology is to identify purely topological conditions that will characterize whether or not a given closed manifold admits a symplectic form, or whether two given symplectic forms are deformation equivalent. For Weinstein manifolds, which are open symplectic manifolds with a bit of extra structure, these problems have known solutions that are elegant and deep. Conjectured strategies to attack the general problem on closed manifolds often make use of the known solution for Weinstein manifolds.
- By a deep result of Donaldson, every closed symplectic manifold X contains distinguished symplectic submanifolds $Y \subset X$ of codimension 2 such that $X \setminus Y$ is naturally a Weinstein manifold.
- Every compact Weinstein manifold, also known as a Weinstein *domain*, has a nonempty boundary that inherits a natural *contact* structure: Weinstein domains thus define a natural class of *symplectic fillings* of closed contact manifolds.
- Symplectic handles define a notion of *contact surgery* on contact manifolds along isotropic submanifolds, such that the contact manifolds before and after surgery are related by a so-called *Weinstein cobordism*. In light of this, one can use symplectic invariants of Weinstein manifolds (such as *symplectic homology* and other Floer-type invariants) to compute related invariants of contact manifolds.
- By a result of Giroux, every closed contact manifold admits an open book decomposition whose binding is a contact submanifold of codimension two, and whose pages carry natural Weinstein structures, making them symplectic fillings of the binding.

The following additional reason for interest in Weinstein manifolds is not purely symplectic, but instead lives at the interface between symplectic and *complex* geometry:

- By a deep theorem originally sketched by Eliashberg (and built on top of a complex-analytical result of Grauert), every Weinstein manifold can be endowed with a complex manifold structure such that it embeds properly and holomorphically into $\mathbb{C}^N$ for some $N \gg 0$. Complex manifolds with this property are called *Stein manifolds*.

Here is a quick overview of the essential notions and results that we aim to understand in this seminar.

Recall that a **symplectic** manifold (W, ω) is an even-dimensional manifold W carrying a symplectic form $\omega \in \Omega^2(W)$, meaning a 2-form that is closed ($d\omega = 0$) and nondegenerate ($\omega(X, \cdot) \neq 0$ for all vectors $X \neq 0$). A **Weinstein structure** on W is a tuple (ω, φ, X) consisting of three pieces of data:

- A symplectic form ω;
- A Morse function $\varphi : W \to \mathbb{R}$, i.e. a smooth function with isolated critical points at which the Hessian is nonsingular;

- A vector field X that satisfies

$$\mathcal{L}_X \omega = \omega \qquad \text{and} \qquad d\varphi(X) > 0 \text{ outside of critical points.}$$

The first condition on X makes it a **Liouville vector field**, so that its flow φ_X^t satisfies $(\varphi_X^t)^* \omega = e^t \omega$ for all t and is thus called a *symplectic dilation*. The second makes it a **gradient-like** vector field with respect to φ.[1] Since ω is nondegenerate, there is a unique 1-form λ determined by the condition

$$\omega(X, \cdot) = \lambda,$$

and by Cartan's magic formula, the fact that X is a Liouville vector field is then equivalent to λ satisfying $d\lambda = \omega$. The symplectic form on a Weinstein manifold is therefore exact, so by Stokes' theorem, W cannot be closed; it is necessarily an open manifold, meaning that each of its connected components is either noncompact or has nonempty boundary. If $\partial W \neq \emptyset$, then we require one further condition, namely

$$d\varphi|_{T(\partial W)} \equiv 0,$$

so that φ is constant on each connected component of ∂W, and the gradient-like Liouville vector field X is therefore transverse to the boundary. One can show that the latter is equivalent to the condition that the 1-form λ restricts to ∂W as a *contact form*, and thus induces a *contact structure* on the boundary (and also on every other regular level set of φ).

The original motivation for Weinstein structures comes from a symplectic variant of classical Morse theory, in which real-valued functions are used to produce handle decompositions, and in this way to understand the topology of smooth manifolds. If W is a compact smooth n-manifold with boundary $M := \partial W$ containing an embedded $(k-1)$-sphere $S^{k-1} \cong L \subset M$ for some $k \in \{1, \dots, n\}$, one can produce a new manifold W' by attaching an n-dimensional **k-handle** along a neighborhood $N(L) \subset M$ of L,

$$W' := W \cup_{N(L)} (\mathbb{D}^k \times \mathbb{D}^{n-k}),$$

where the attachment identifies $S^{k-1} \times \mathbb{D}^{n-k} \cong N(L) \subset M$ with $\partial \mathbb{D}^k \times \mathbb{D}^{n-k} \subset \partial(\mathbb{D}^k \times \mathbb{D}^{n-k})$. The boundary $M' := \partial W'$ of the new manifold constructed in this way can be described as the result of a surgery operation on M, in which $N(L) \cong S^{k-1} \times \mathbb{D}^{n-k}$ is replaced by a copy of $\mathbb{D}^k \times \partial \mathbb{D}^{n-k} = \mathbb{D}^k \times S^{n-k-1}$ using the obvious identification

$$\partial(S^{k-1} \times \mathbb{D}^{n-k}) = S^{k-1} \times S^{n-k-1} = \partial(\mathbb{D}^k \times S^{n-k-1}).$$

The k-dimensional disk $\mathbb{D}^k \times \{0\} \subset \mathbb{D}^k \times \mathbb{D}^{n-k}$ is called the **core** of the handle, whose boundary gets identified with the so-called **attaching sphere** $S^{k-1} \cong L \subset M$. Dually, the new manifold M' contains a distinguished $(n-k-1)$-sphere called the **belt sphere**, which is the boundary of the **cocore** $\{0\} \times \mathbb{D}^{n-k} \subset \mathbb{D}^k \times \mathbb{D}^{n-k}$ of the handle. Note that this construction also makes sense for $k = 0$ if W is taken to be the empty set: attaching an n-dimensional 0-handle thus means replacing the empty set with a disk $\mathbb{D}^n$. Dual to this, if $M = \partial W \cong S^{n-1}$, then one can attach an n-handle along the attaching sphere $L := M$, which means capping off ∂W with a disk to produce a closed manifold.

In classical Morse theory as described in [Mil63], each Morse function $\varphi : W \to \mathbb{R}$ on a smooth n-manifold W gives rise to a handle decomposition of W, in which there is a one-to-one correspondence between k-handles for $k \in \{0, \dots, n\}$ and critical points of φ with index k, meaning points at which φ can be written in the form

$$\varphi(x_1, \dots, x_n) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2 + \text{const}$$

in suitable coordinates. For reasonable choices of gradient-like vector fields X, the **stable manifold** and **unstable manifold** of each critical point—meaning the sets of points that asymptotically approach the critical point or emerge from it respectively under the flow of X—can be regarded as the core and cocore respectively of the corresponding handle.

[1]Strictly speaking, the most useful definition of the term "gradient-like" also requires X and φ to satisfy a technical condition at the critical points of φ, which is explained in [CE12], though this detail is often ignored in papers on the subject.

The symplectic version of this story begins with an observation originally due to Weinstein [Wei91]: for each $k = 0, \ldots, n$, one can define $2n$-dimensional **symplectic k-handles**

$$(\mathbb{D}^k \times \mathbb{D}^{2n-k}, \omega, \varphi, X),$$

equipped with not only a symplectic form ω but also a Morse function φ having a unique critical point of index k at $(0,0) \in \mathbb{D}^k \times \mathbb{D}^{2n-k}$, whose gradient $X := \nabla\varphi$ is a Liouville vector field pointing transversely inward at $\partial\mathbb{D}^k \times \mathbb{D}^{2n-k}$ and outward at $\mathbb{D}^k \times \partial\mathbb{D}^{2n-k}$. Symplectic handles thus come naturally with Weinstein structures, and they can be used as basic building blocks to construct Weinstein manifolds by handle attachment. Conversely, any Weinstein manifold (W, ω, φ, X) can be decomposed into symplectic handles that are in bijective correspondence with the critical points of the Morse function φ.

You may have missed the following detail in the above discussion: a $2n$-dimensional symplectic k-handle can be defined for each $k = 0, \ldots, n$, but *not* for $k = n+1, \ldots, 2n$. There are good geometric reasons for this restriction: the core $C := \mathbb{D}^k \times \{0\} \subset \mathbb{D}^k \times \mathbb{D}^{2n-k}$ of a symplectic handle is an **isotropic** submanifold (meaning $\omega|_{TC} \equiv 0$), as is the stable manifold of any critical point in a Weinstein manifold, while dually, cocores of symplectic handles and unstable manifolds of critical points in Weinstein manifolds are **coisotropic** submanifolds. By basic symplectic linear algebra, isotropic submanifolds in a symplectic $2n$-manifold can have dimension at most n; this is why symplectic k-handles are only possible for $k \le n$. It means there is a significant topological restriction on the $2n$-dimensional manifolds that admit Weinstein structures: such a manifold must admit a Morse function whose critical points all have index at most n, making it retractible to an n-dimensional CW-complex, a property that $2n$-manifolds do not typically have. It is a remarkable fact—and one of the main results that we will study in this seminar—that for $n > 2$, the existence of such a Morse function is not just necessary, but very nearly also *sufficient*:

Theorem. *For $n > 2$, a $2n$-dimensional manifold W admits a Weinstein structure if and only if it admits a nondegenerate 2-form (or equivalently an almost complex structure) and an exhausting Morse function whose critical points all have index at most n.*

This theorem is a *flexibility* result, also known as an *h-principle*, because it says that the obvious minimal topological conditions necessary for the existence of a certain geometric structure (in this case a Weinstein structure) are also sufficient.

The picture becomes more remarkable with a bit of complex geometry added into the mix. Classically, a **Stein manifold** is a (necessarily noncompact) complex manifold (W, J) that admits a proper holomorphic embedding into $\mathbb{C}^N$ for some $N \gg 0$. Here, we denote by $J : TW \to TW$ the associated almost complex structure, which defines multiplication by i on the complex tangent spaces, but we stress that J is assumed to be an *integrable* almost complex structure, meaning it comes from an atlas of complex coordinate charts whose transition maps are holomorphic. For any properly and holomorphically embedded Stein manifold $W \subset \mathbb{C}^N$, restricting the function $\varphi : \mathbb{C}^n \to \mathbb{R} : z \mapsto |z|^2$ to W produces a smooth function $\varphi : W \to \mathbb{R}$ with the following two properties:

- It is **exhausting**, i.e. it is proper and bounded below;
- It is **plurisubharmonic**, also called **J-convex**, which means that the 2-form

  $$\omega_\varphi := -d(d\varphi \circ J) \in \Omega^2(W)$$

  satisfies $\omega_\varphi(X, JX) > 0$ for all vectors $X \neq 0$.[2] In particular, ω_φ is a symplectic form that *tames* the complex structure of W.

By a complex-analytical result of Grauert (which sadly goes beyond the scope of this seminar), the existence of such a function characterizes Stein manifolds: a complex manifold (W, J) is a Stein manifold *if and only if* it admits an exhausting J-convex function $\varphi : W \to \mathbb{R}$. Since J-convexity is an open condition, φ can

[2] Plurisubharmonicity can also be described as the condition that φ restricts to a strictly subharmonic function on every complex curve.

also be assumed to be a Morse function without loss of generality. It is not difficult to show that in this situation, the gradient $X := \nabla\varphi$ with respect to the Riemannian metric $g := \omega_\varphi(\cdot, J\cdot)$ is a Liouville vector field on the symplectic manifold (W, ω_φ), thus making $(\omega_\varphi, \varphi, \nabla\varphi)$ a Weinstein structure on W. Here is the result that inspires the title of [CE12]:

Theorem. *Every Weinstein structure is homotopic to one that arises in the manner described above from a Stein structure. Moreover, two Stein structures are homotopic as Weinstein structures if and only if they are homotopic as Stein structures.*

Corollary. *For $n > 2$, a $2n$-manifold can be made into a Stein manifold if and only if it admits both an almost complex structure and an exhausting Morse function whose critical points all have index at most n.*

Note that in general, it is considered a quite difficult problem to determine whether a given smooth (real) manifold of even dimension can be made into a complex manifold—the case of S^6 for example is famously open.[3] The corollary above shows however that this problem becomes tractable, and has a quite simple solution in the language of algebraic topology, as soon as one adds in a bit of extra data beyond the complex structure.

A final important result worth mentioning is that for $n > 2$, there is also a special class of Weinstein structures called **flexible**, whose classification up to homotopy satisfies an h-principle. The flexible Weinstein structures notably include those which are **subcritical**, meaning that the Morse function φ has only critical points with indices *strictly* less than n. Emmy Murphy's discovery [Mur] of **loose Legendrian submanifolds** in 2012 revealed that there are also flexible Weinstein structures containing critical handles, and in fact, every Weinstein structure is homotopic via nondegenerate 2-forms to a *unique* one that is flexible:

Theorem. *Two flexible Weinstein structures on a manifold W of dimension $2n > 4$ are homotopic as Weinstein structures if and only if their underlying nondegenerate 2-forms are homotopic as nondegenerate 2-forms (or equivalently, almost complex structures).*

For all three of the theorems stated above, the general idea behind the proof is to work with handle decompositions, developing symplectic analogues of constructions that originate in classical Morse theory. This includes some constructions that first appeared in the proof of Smale's h-cobordism theorem, on which the solution to the higher-dimensional Poincaré conjecture was based.

Literature

The seminar will be based almost entirely on the book [CE12] by Cieliebak and Eliashberg, with other sources referred to occasionally as needed. The book contains strictly *too much* material; we will focus mostly on aspects of the story that are *symplectic* rather than *complex*, though we will touch upon complex geometry as well.

Plan of talks

1. October 17: *Introduction and planning of further talks*
   I will give a general introduction to the subject, including much of what is discussed above and some essential definitions. We will then distribute future talks among the participants. (The exact details of how we do this will depend on how many participants there are, but I plan on fixing speakers for at least the next three talks at this meeting.)

[3] An obvious necessary condition for a real manifold to be made into a complex manifold is that it must admit an *almost* complex structure, a condition that can be characterized completely in algebro-topological terms using characteristic classes and obstruction theory. For S^6, it is easy to write down an explicit almost complex structure, but easy examples one can write down are not integrable. The question of which almost complex manifolds can actually be made into complex manifolds is, in general, wide open.

2. October 24: *Notions from complex geometry* (Chapter 5, plus a few definitions from Chapter 2)
   We're mostly skipping Chapters 2–4 of Cieliebak-Eliashberg because they mostly consist of definitions (which can be introduced later as needed) and technical lemmas that are not very deep and not so hard to believe (so we will take them as black boxes). The essential things in Chapter 5 are the notions of J-convexity and holomorphic convexity, and real-analytic approximation (§5.7 and §5.8). The rest of the chapter contains a lot of material that is inessential for the seminar, but can serve as an interesting survey of ideas from complex geometry for those who are unfamiliar with it; which of those topics you include is a matter of taste.

3. October 31: *Symplectic and contact preliminaries* (Chapter 6)
   This is mostly standard basic stuff from symplectic geometry: the essential things to cover are symplectic/isotropic/coisotropic/Lagrangian subspaces and submanifolds, compatible (almost) complex structures and real subspaces/submanifolds, contact manifolds and Legendrian submanifolds, the Moser deformation trick and the various neighborhood theorems (both symplectic and contact), and Corollary 6.25 on real-analytic approximation. The contents of §6.8 are interesting but can be considered inessential.

4. November 7: *h-principles* (Chapter 7)
   It would make sense for this talk to be given by someone who attended last semester's seminar on the h-principle: the goal is to explain the statements of several nontrivial flexibility results that were covered in that seminar, plus one or two that were not. One can typically present these results in the form "formal solutions are homotopic to genuine solutions" without needing to discuss any general jet-space formalism. For results whose proofs do not appear in Cieliebak-Eliashberg, we will be content to take them as black boxes in the seminar. In particular, Murphy's h-principle for loose Legendrians, covered in §7.7, is an important result that must be stated, but its proof was the main result of a Ph.D. thesis, so it would not be practical to present it here.

5. November 14: no seminar

6. November 21: no seminar

7. November 28: *Ideas from Morse-Smale theory* (Chapter 9)
   The goal of this talk is to outline enough of the main ideas from classical Morse theory to explain (at least in intuitive terms) the proof of the h-cobordism theorem sketched in §9.8, followed by the two-index theorem in §9.9. The really important ideas in this chapter appear from §9.6 onwards; what comes before that is mainly technical results that are neither difficult nor deep, and most of them can be stated without proof.

8. December 5: *Liouville and Weinstein structures* (Chapter 11)
   A few of the important definitions in this chapter will have already appeared in the first talk and can thus be skipped. The notions of Weinstein/Stein *homotopies* in §11.6 are a bit subtle (at least when working on noncompact manifolds), and will require some attention. Otherwise, the most important result here is Prop. 11.25, on Weinstein and Stein structures with a unique critical point.

9. December 12: *Modifications of Weinstein structures* (Chapter 12)
   The most important things in this chapter are the last two sections on Morse-Smale theory and elementary Weinstein homotopies, so everything else should be seen as stepping stones on the way to those topics. Many of the proofs are technical but not deep, so it's fine to omit some of them due to lack of time.

10. December 19: no seminar

11. January 9: no seminar

12. January 16: *Existence of Stein and Weinstein structures* (Chapter 13 and some of Chapter 8)
    The goal here is to assemble all the technical results established so far in order to prove the main theorems on existence of Weinstein structures in higher dimensions and the fact that Weinstein structures can always be deformed to Stein structures. You can mostly follow Chapter 13 for this, but you

will also need to dip into §8.3 for the Gromov-Landweber theorem on existence of complex structures, which uses some of the material from Chapter 5 on real-analyticity. I would recommend omitting the rest of Chapter 8 as much as possible, since most of the results proved there can also be deduced from the results in Chapter 13 about Weinstein structures.

13. January 23: *Deformations of Weinstein and Stein structures* (Chapters 14 and 15)
These two chapters complete the proof of the other main theorem in the book, showing the equivalence of Stein homotopies and Weinstein homotopies, so that the natural map sending Stein structures to Weinstein structures induces a bijection on π_0. Another important topic addressed in Chapter 14 is the h-principle for *flexible* Weinstein structures, which is important and deep, but you should consider it a secondary priority after the main theorem about Stein/Weinstein deformations. The contents of §14.4 and (to a lesser extent) §14.5 are also cute, but should be covered only if time permits.

14. January 30: *Dimension four* (Chapter 16)
This chapter is a departure from the rest of the book, as it describes *rigidity* (rather than *flexibility*) results that can be proved about 4-dimensional Stein domains using holomorphic curve methods. One important corollary of these results is a counterexample to the $n = 2$ case of the theorem on existence of Weinstein structures: $S^2 \times \mathbb{R}^2$ admits both an almost complex structure and an exhausting Morse function with critical points of index at most 2, but it does not admit a Weinstein structure—if it did, then it would produce a symplectic filling of $S^2 \times S^1$ that holomorphic curve results show to be impossible. The talk is best given by someone who is already familiar with holomorphic curves, and the basic technical results about them can be presented as black boxes when needed. The proofs described in the chapter are mostly expansions on the *original* proofs that were sketched by Eliashberg around 1990, using holomorphic disks, but in many cases, I would say that there are nicer ways to prove those results using more modern technology out of symplectic field theory. Ask me if you want details.

15. February 6: *Exotic Stein structures* (Chapter 17)
This chapter sketches another class of rigidity theorems based on holomorphic curve methods (in this case Symplectic Homology), leading to the beautiful result that for any higher-dimensional manifold admitting a finite-type Stein structure, there are in fact infinitely-many distinct Stein homotopy classes of Stein structures that are homotopic as almost complex structures. As a special case, this is one way to obtain exotic symplectic structures on $\mathbb{R}^{2n}$ for $n > 2$. The goal here is only to give an outline of the ideas, since it would not be practical to fully develop the technology needed for these proofs. For more details on the constructions, one can consult the papers by McLean, Seidel-Smith and/or Abouzaid-Seidel cited in the book.

16. February 13: *To be decided*

Otherwise

All other practical information such as the location and an up-to-date schedule for the seminar is posted on the seminar webpage at

http://www.mathematik.hu-berlin.de/~wendl/Winter2025/Weinstein/

References

[CE12] K. Cieliebak and Y. Eliashberg, *From Stein to Weinstein and back: symplectic geometry of affine complex manifolds*, American Mathematical Society Colloquium Publications, vol. 59, American Mathematical Society, Providence, RI, 2012.

[Mil63] J. W. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963.

[Mur] E. Murphy, *Loose Legendrian embeddings in high dimensional contact manifolds*. Preprint arXiv:1201.2245.

[Wei91] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. **20** (1991), no. 2, 241–251.