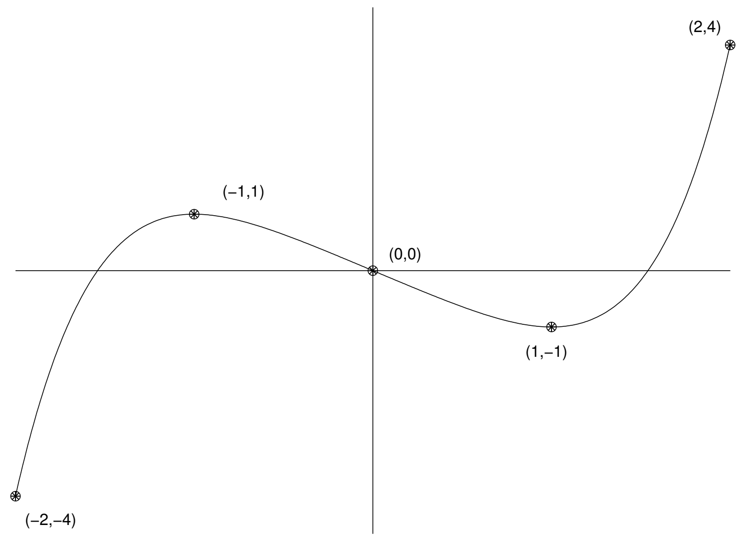
MIDTERM 2 - 04/10/03
Calculus I - V63.0121

Masmoudi–Schneiderman–Wendl

1. (10pts.) Let $F(t) = \ln(t-2) + e^{-t}$. Find $F'(t)$ and $F''(t)$. ANSWER: $F'(t) = \frac{1}{t-2} - e^{-t}$, $F''(t) = -\frac{1}{(t-2)^2} + e^{-t}$.

2. (10pts.) Sketch the graph of a continuous function $g(x)$ defined on the closed interval $[-2, 2]$ such that g has an absolute minimum at $x = -2$, a local maximum at $x = -1$, an inflection point at $x = 0$, a local minimum at $x = 1$ and an absolute maximum at $x = 2$.



3. (10pts.) Find $\frac{dy}{dx}$ in terms of x and y if $3y^5 - 2x^3 = 2003$. ANSWER: $\frac{d}{dx}3y^5 - \frac{d}{dx}2x^3 = \frac{d}{dx}2003 \Rightarrow 15y^4\frac{dy}{dx} - 6x^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{6x^2}{15y^4} = \frac{2x^2}{5y^4}$.

4. (10pts.) Find the absolute maximum and absolute minimum of $f(x) = \sin(x) + \cos(x)$ on the closed interval $[0, \pi]$. ANSWER: $f'(x) = \cos(x) - \sin(x) = 0$ for $0 \le x \le \pi$ only at $x = \pi/4$. Comparing $f(\pi/4) = \sqrt{2}$ with the endpoints $f(0) = 1$, $f(\pi) = -1$, we see that the absolute minimum is -1 (at $x = \pi$), and the maximum is $\sqrt{2}$ (at $x = \pi/4$).

5. (15pts.) Find the dimensions of a rectangle with area 100 whose perimeter is as small as possible. ANSWER: Let x be the width of the rectangle and let y be the height. Then the area is $100 = xy$, and the perimeter is $2x + 2y$. Rewrite this as a function of x alone by substituting $y = 100/x$: $P(x) = 2x + 200/x$. We want to find the absolute minimum of $P(x)$ for $x > 0$. $P'(x) = 2 - 200/x^2 = 0 \Rightarrow x^2 = 100 \Rightarrow x = 10$ since x must be positive (therefore not -10). Observe that $P'(x) < 0$ for $0 < x < 10$ (e.g. plug in $x = 1$) and $P'(x) > 0$ for $x > 10$ (e.g. plug in $x = 20$). Thus $x = 10$ is a local minimum, and since there are no other critical points, it's the only one. Finally, $y = 100/x = 100/10 = 10$. So the rectangle is a square of side 10.

6. (15pts.) Find an equation for the tangent line to the curve $\sin y = x$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$. ANSWER: The equation will be $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = \left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $m = \frac{dy}{dx}$ at that point. By implicit differentiation, $\frac{d}{dx}\sin y = \frac{d}{dx}x \Rightarrow \cos y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\pi/4)} = \sqrt{2}$. So the equation is $y - \frac{\pi}{4} = \sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right)$.

7. (10pts.) Show that the equation $x^4+4x+1=0$ has at most two real roots. ANSWER: Let $f(x)=x^4+4x+1$. If a and b are two points where $f(a)=f(b)=0$, then by Rolle's theorem there must be at least one critical point somewhere between a and b. Thus if there are three such points, there are at least two critical points (at least one between each pair of roots). But $f'(x)=4x^3+4=4(x^3+1)=0$ only when $x=-1$; since there is only one critical point, there can't be more than two roots.

8. (20pts.) Sketch the graph of $G(x)=x^2-\dfrac{2}{x}$ indicating the local maxima and minima, inflection points, asympototes and limits at $\pm\infty$. ANSWER: The function is not continuous at $x=0$, and there's a vertical asymptote there since $\frac{2}{x}$ becomes infinite as $x\to 0$. As $x\to\pm\infty$, $\frac{2}{x}\to 0$ but $x^2\to\infty$, so there's no horizontal asymptote. $G'(x)=2x+\frac{2}{x^2}=2\left(x+\frac{1}{x^2}\right)=0$ only if $x+\frac{1}{x^2}=0\Rightarrow x^3+1=0\Rightarrow x=-1$. So -1 is the only critical point, and we can check that $G'(x)<0$ (G is decreasing) for $x<-1$, while $G'(x)>0$ (G is increasing) for $x>-1$. $G''(x)=2-\frac{4}{x^3}=2\left(1-\frac{2}{x^3}\right)=0\Rightarrow x^3=2\Rightarrow x=\sqrt[3]{2}\approx 1.26\ldots$, so $\sqrt[3]{2}$ is the only inflection point, and we find $G''(x)>0$ (G is concave up) for $x<0$ or $x>\sqrt{2}$, while $G''(x)<0$ (G is concave down) for $0<x<\sqrt{2}$. We should plot the critical point and the inflection point: $G(-1)=(-1)^2-\frac{2}{-1}=3$ and $G(\sqrt[3]{2})=(\sqrt[3]{2})^2-\frac{2}{\sqrt[3]{2}}=2^{2/3}-2^{2/3}=0$. See below.