Calculus II
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Integration Review Problems

If you need to review integration techniques from Calculus I, the following problems (and accompanying solutions) should help. The majority can be solved by an intelligent choice of substitution; you will not need integration by parts or partial fractions. At least one problem can be done by trigonometric substitution, but it can also be solved geometrically.

1. $\displaystyle\int_1^3 \frac{4}{x^2}\,dx$

2. $\displaystyle\int_1^3 \frac{dx}{(x+2)^2}$

3. $\displaystyle\int_1^3 \frac{dx}{(x-2)^2}$

4. $\displaystyle\int \frac{6x^2-4}{x^3-2x+4}\,dx$

5. $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\,d\theta$

6. $\displaystyle\int_0^{\pi} \frac{\sin\theta}{\cos^2\theta}\,d\theta$

7. $\displaystyle\int_0^1 \sin(\pi t)\cos(\cos\pi t)\,dt$

8. $\displaystyle\int_{1/2}^x \sin(\pi t)\cos(\cos\pi t)\,dt$

9. $\displaystyle\frac{d}{dx}\left(\int_{1/2}^x \sin(\pi t)\cos(\cos\pi t)\,dt\right)$

10. Let $f(x) = \displaystyle\int_{x^2/\pi}^0 \sin(\pi\theta)\cos(\cos\pi\theta)\,d\theta$. Find $f'(x)$.

11. $\displaystyle\frac{d}{dx}\left(\int_0^1 \sin(\pi x)\cos(\cos\pi x)\,dx\right)$

12. $\displaystyle\frac{d}{dt}\left(\int_{\sin t}^0 \arcsin x\,dx\right)$

13. $\displaystyle\int_0^{\sqrt{2}/2} \frac{d}{dy}(\arcsin y)\,dy$

14. $\displaystyle\int_0^{\arcsin t} \frac{d}{dx}e^{\sin x}\,dx$

15. $\displaystyle\int_0^2 -\sqrt{4-(x-2)^2}\,dx$

16. $\displaystyle\int e^{3x}\,dx$

17. $\displaystyle\int_1^2 e^{2\ln x}\,dx$

18. $\displaystyle\int_e^{e^2} \frac{(\ln x)^2}{x}\,dx$

19. $\displaystyle\int_{-1}^1 x^2 e^{x^3}\,dx$

20. $\displaystyle\int_{-2}^2 x\,dx$

21. $\displaystyle\int_{-2}^2 x^2\,dx$

22. $\displaystyle\int_{1/2}^{1/2} \cos(e^x)\,dx$

23. Let $F(x) = \dfrac{1}{2}\displaystyle\int_{x^2-1}^{0} \frac{(t^2-1)(t+2)}{\cos \pi t}\,dt + e^{\pi t}$.
Find an equation for the tangent line to the curve $y = F(x)$ at $x = -1$.

24. Let $g(x) = x^2 - \displaystyle\int_0^{2x} \frac{t}{\sqrt{t^2+5}}\,dt$.
Find an equation for the tangent line to the curve $y = g(x)$ at $x = 1$.

25. Let

$$F(x) = \begin{cases} x+3 & x \le -2, \\ 1+\sqrt{1-(x+1)^2} & -2 \le x \le 0, \\ 1-\sqrt{x} & x \ge 0. \end{cases}$$

Find $\int_{-2}^0 F(x)\,dx$, $\int_{-3}^0 F(x)\,dx$ and $\int_{-3}^1 F(x)\,dx$.

26. $\displaystyle\int_0^{\pi/4} \frac{(\cos^3 x - 1)^2}{\cos^2 x}\sin x\,dx$

27. Prove that $\displaystyle\int_0^2 e^{\sqrt{x^2-2x+5}}\,dx \ge 2e^2$.
(Hint: what is the minimum of $x^2 - 2x + 5$ on the given interval?)

28. $\dfrac{d}{dx}\sqrt{\ln(1-x^2)}$

29. Let

$$f(x) = \begin{cases} (x-2)^2 & x \le 2 \\ \sqrt{x-2} & x \ge 2 \end{cases}$$

Find $f'(0)$, $f'(6)$ and $\int_0^4 f(x)\,dx$.

30. $\left|\displaystyle\int_0^4 (x^2 - 5x + 6)\,dx\right|$

31. $\displaystyle\int_0^4 |x^2 - 5x + 6|\,dx$
(Hint: find out when $x^2 - 5x + 6$ is positive and when it's negative. Then you can define the function "piecewise" as in problem 29.)

32. $\displaystyle\int_{-2}^2 |x^2 - 1|\,dx$.

Solutions

1. $\displaystyle\int_1^3 \frac{4}{x^2}\,dx = 4\int_1^3 x^{-2}\,dx = 4\left(\frac{x^{-1}}{-1}\right)\Big|_1^3 = -\frac{4}{x}\Big|_1^3 = -\frac{4}{3} - \left(-\frac{4}{1}\right) = \frac{8}{3}.$

2. $\displaystyle\int_1^3 \frac{dx}{(x+2)^2}$. Substitute $u = x+2$, $\frac{du}{dx} = 1$:

$$\int_1^3 \frac{1}{(x+2)^2}\,dx = \int_1^3 \frac{1}{u^2}\frac{du}{dx}\,dx = \int_{u(1)}^{u(3)} \frac{1}{u^2}\,du = \int_3^5 u^{-2}\,du = -u^{-1}\Big|_3^5 = -\frac{1}{u}\Big|_3^5 = -\frac{1}{5} - \left(-\frac{1}{3}\right) = \frac{2}{15}.$$

3. $\displaystyle\int_1^3 \frac{dx}{(x-2)^2}$. The correct answer is **undefined**, because $\frac{1}{(x-2)^2}$ is not continuous on the interval from 1 to 3 (it's undefined at $x = 2$). If you didn't notice that, you might have tried to solve this by substituting $u = x - 2$ and computing the integral with the fundamental theorem the same as in problem 2. The answer you get by this method is -8, but we can tell by looking at the function, whose value is always *positive*, that its integral over any interval should never be negative. So in this case the fundamental theorem gives an obviously wrong answer, because the function is not continuous everywhere between 1 and 3. Pay attention!

4. $\displaystyle\int \frac{6x^2-4}{x^3-2x+4}\,dx$. What we're asked for here is an *antiderivative*, not an integral, so there's no need to worry about whether the function is continuous on any particular interval; we simply need to find a function whose derivative is $\frac{6x^2-4}{x^3-2x+4}$. Try the substitution $u = x^3 - 2x + 4$, with $\frac{du}{dx} = 3x^2 - 2$, and notice that $2\frac{du}{dx} = 6x^2 - 4$ is the numerator in the integrand. So,

$$\int \frac{6x^2-4}{x^3-2x+4}\,dx = \int \frac{2\,du/dx}{u}\,dx = 2\int \frac{1}{u}\frac{du}{dx}\,dx = 2\int \frac{1}{u}\,du = 2\ln|u| = 2\ln|x^3-2x+4|.$$

5. $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\,d\theta$. We first must check that the function in the integrand is continuous on the interval from 0 to $\frac{\pi}{4}$: clearly $\frac{\sin\theta}{\cos^2\theta}$ is continuous as long as $\cos\theta \neq 0$, and this is true when $0 \leq \theta \leq \frac{\pi}{4}$. Thus the integral is defined and we can solve it using the fundamental theorem. Try the substitution $u = \cos\theta$, $\frac{du}{d\theta} = -\sin\theta$:

$$\int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} \frac{-du/d\theta}{u^2}\,d\theta = -\int_0^{\pi/4} \frac{1}{u^2}\frac{du}{d\theta}\,d\theta = -\int_{u(0)}^{u(\pi/4)} u^{-2}\,du = \frac{1}{u}\Big|_1^{\sqrt{2}/2} = \sqrt{2} - 1.$$

6. $\displaystyle\int_0^{\pi} \frac{\sin\theta}{\cos^2\theta}\,d\theta$. Again the first step is to check that the function is continuous: notice that $\cos(\pi/2) = 0$, thus $\frac{\sin\theta}{\cos^2\theta}$ is not defined at $\theta = \pi/2$, and it cannot be continuous there. So the answer is **undefined**. This was not the case in problem 5 because $\pi/2$ was not part of the interval from 0 to $\pi/4$. Blindly applying the fundamental theorem in this case would lead to the answer -2, which we can see must be wrong because $\frac{\sin\theta}{\cos^2\theta} \geq 0$ when $0 \leq \theta \leq \pi$.

7. $\displaystyle\int_0^1 \sin(\pi t)\cos(\cos\pi t)\,dt$. Substitute $u = \cos\pi t$, $\frac{du}{dx} = -\pi\sin\pi t$:

$$\int_0^1 \sin(\pi t)\cos(\cos\pi t)\,dt = \int_0^1 \left(-\frac{1}{\pi}\frac{du}{dx}\right)\cos u\,dt = -\frac{1}{\pi}\int_0^1 \cos u\frac{du}{dt}\,dt = -\frac{1}{\pi}\int_{u(0)}^{u(1)} \cos u\,du$$

$$= -\frac{1}{\pi}\sin u\Big|_1^{-1} = -\frac{1}{\pi}[\sin(-1) - \sin(1)] = -\frac{1}{\pi}[-\sin(1) - \sin(1)] = \frac{2}{\pi}\sin(1) \approx 0.536.$$

8. $\int_{1/2}^{x} \sin(\pi t)\cos(\cos \pi t)\, dt$. Using the same substitution as in problem 7, $u = \cos \pi t$,

$$\int_{1/2}^{x} \sin(\pi t)\cos(\cos \pi t)\, dt = -\frac{1}{\pi}\sin u\Big|_{u(1/2)}^{u(x)} = -\frac{1}{\pi}\sin u\Big|_{0}^{\cos \pi x} = -\frac{1}{\pi}\sin(\cos \pi x).$$

9. $\frac{d}{dx}\left(\int_{1/2}^{x} \sin(\pi t)\cos(\cos \pi t)\, dt\right) = \sin \pi x \cos(\cos \pi x)$. This follows immediately from part 1 of the fundamental theorem of calculus, and you'll find you get the same result if you differentiate the answer to problem 8 with respect to x (try it if you're skepticle).

10. Let $f(x) = \int_{x^2/\pi}^{0} \sin(\pi\theta)\cos(\cos \pi\theta)\, d\theta$. Find $f'(x)$. The first thing to do is invert the limits so that the term with x in it is the upper limit of integration:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_{x^2/\pi}^{0} \sin(\pi\theta)\cos(\cos \pi\theta)\, d\theta \\ &= \frac{d}{dx}\left(-\int_{0}^{x^2/\pi} \sin(\pi\theta)\cos(\cos \pi\theta)\, d\theta\right) = -\frac{d}{dx}\int_{0}^{x^2/\pi} \sin(\pi\theta)\cos(\cos \pi\theta)\, d\theta. \end{aligned}$$

To compute any derivative of an integral in this form we use part 1 of the fundamental theorem, together with the chain rule: set $u = x^2/\pi$, then

$$\begin{aligned} &\frac{d}{dx}\int_{0}^{x^2/\pi} \sin(\pi\theta)\cos(\cos \pi\theta)\, d\theta = \frac{d}{dx}\int_{0}^{u} \sin(\pi\theta)\cos(\cos \pi\theta)\, d\theta \\ &= \left(\frac{d}{du}\int_{0}^{u} \sin(\pi\theta)\cos(\cos \pi\theta)\, d\theta\right)\cdot\frac{du}{dx} = [\sin(\pi u)\cos(\cos \pi u)]\left(\frac{2x}{\pi}\right) = \frac{2x}{\pi}\sin\left(x^2\right)\cos\left[\cos\left(x^2\right)\right]. \end{aligned}$$

Thus $f'(x) = -\frac{2x}{\pi}\sin\left(x^2\right)\cos\left[\cos\left(x^2\right)\right]$.

11. $\frac{d}{dx}\left(\int_{0}^{1} \sin(\pi x)\cos(\cos \pi x)\, dx\right) = 0$ because any integral with constant limits, once evaluated, has no dependence on x, i.e. it's a constant, and the derivative of a constant is 0. (Compare with problems 9 and 10, in which the limits of integration *are* dependent on x.)

12. $\frac{d}{dt}\left(\int_{\sin t}^{0} \arcsin x\, dx\right) = -\frac{d}{dt}\left(\int_{0}^{\sin t} \arcsin x\, dx\right) = -\arcsin(\sin t)\cdot\frac{d}{dt}\sin t = -t\cos t.$
In the last step we've used the fact that $\arcsin(\sin t) = t$.

13. $\int_{0}^{\sqrt{2}/2} \frac{d}{dy}(\arcsin y)\, dy = \arcsin y\Big|_{0}^{\sqrt{2}/2} = \frac{\pi}{4}$. This is a direct application of the fundamental theorem, part 2: $\arcsin y$ is the *antiderivative* of $\frac{d}{dy}(\arcsin y)$.

14. $\int_{0}^{\arcsin t} \frac{d}{dx}e^{\sin x}\, dx = e^{\sin x}\Big|_{0}^{\arcsin t} = e^{\sin(\arcsin t)} - e^{\sin(0)} = e^t - 1.$

15. $\int_{0}^{2} -\sqrt{4-(x-2)^2}\, dx$. This integral must be computed geometrically: if $y = -\sqrt{4-(x-2)^2}$ then $y^2 = 4-(x-2)^2$, and thus $(x-2)^2 + y^2 = 4$. This is the equation of a circle with radius 2, centered at $(2,0)$. The function $-\sqrt{4-(x-2)^2}$ represents only the bottom half of this circle (it cannot be both halves, since then the function would have two values at most x), and the interval from 0 to 2 contains

the left half of the circle. Thus the integral is equal to negative the area of the bottom-left quarter of the circle:

$$\int_0^2 -\sqrt{4-(x-2)^2}\,dx = -\frac{1}{4}\pi(2^2) = -\pi.$$

16. $\int e^{3x}\,dx$. Substitute $u = 3x$, $\frac{du}{dx} = 3$:

$$\int e^{3x}\,dx = \int e^u \left(\frac{1}{3}\frac{du}{dx}\right) dx = \frac{1}{3}\int e^u \frac{du}{dx}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u = \frac{1}{3}e^{3x}.$$

17. $\int_1^2 e^{2\ln x}\,dx = \int_1^2 \left(e^{\ln x}\right)^2\,dx = \int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{7}{3}$. This is why it's useful to keep in mind some basic identities involving exponentials: $e^{\ln x} = x$ and $(e^x)^y = e^{xy}$.

18. $\int_e^{e^2} \frac{(\ln x)^2}{x}\,dx$. Don't forget to confirm first that the function is defined and continuous on the given interval: in this case there's no problem when $e \le x \le e^2$, since the denominator is nonzero and $\ln x$ is defined and continuous for all positive x. Now substitute $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$:

$$\int_e^{e^2} \frac{(\ln x)^2}{x}\,dx = \int_e^{e^2} u^2\frac{du}{dx}\,dx = \int_{\ln e}^{\ln(e^2)} u^2\,du = \int_1^2 u^2\,du = \left.\frac{u^3}{3}\right|_1^2 = \frac{7}{3}.$$

19. $\int_{-1}^1 x^2 e^{x^3}\,dx$. Substitute $u = x^3$, $\frac{du}{dx} = 3x^2$:

$$\int_{-1}^1 x^2 e^{x^3}\,dx = \int_{-1}^1 \left(\frac{1}{3}\frac{du}{dx}\right) e^u\,dx = \frac{1}{3}\int_{-1^3}^{1^3} e^u\,du = \left.\frac{1}{3}e^u\right|_{-1}^1 = \frac{1}{3}\left(e - \frac{1}{e}\right) \approx 0.784.$$

20. $\int_{-2}^2 x\,dx = \left.\frac{x^2}{2}\right|_{-2}^2 = 2 - 2 = 0$. This answer can also be deduced without computation: since $f(x) = x$ is an odd function (i.e. $f(-x) = -f(x)$), the integral from -2 to 0 cancels out the integral from 0 to 2.

21. $\int_{-2}^2 x^2\,dx = \left.\frac{x^3}{3}\right|_{-2}^2 = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$. CAREFUL! In case you were tempted to use the argument of problem 20 and say that this is 0, The argument of problem 20 does not apply here because $f(x) = x^2$ is *not* an odd function, in fact the integral from -2 to 0 has the same value and sign as the integral from 0 to 2. They do not cancel each other out.

22. $\int_{1/2}^{1/2} \cos(e^x)\,dx = 0$, because the upper and lower limits of integration are the same.

23. Let $F(x) = \frac{1}{2}\int_{x^2-1}^0 \frac{(t^2-1)(t+2)}{\cos \pi t}\,dt + e^{\pi x}$.
Find an equation for the tangent line to the curve $y = F(x)$ at $x = -1$. As in any tangent line problem, the answer will be an equation of the form

$$y - y_0 = m(x - x_0),$$

where (x_0, y_0) is any point on the line, and m is its slope. Here it is appropriate to choose $x_0 = -1$, in which case $y_0 = F(-1)$ and $m = F'(-1)$, and the problem is now to compute these two numbers:

$$F(-1) = \frac{1}{2}\int_0^0 \frac{(t^2-1)(t+2)}{\cos \pi t}\,dt + e^{-\pi} = 0 + e^{-\pi} = \frac{1}{e^\pi}.$$

And,

$$F'(x) = \frac{1}{2}\frac{d}{dx}\left(\int_{x^2-1}^{0} \frac{(t^2-1)(t+2)}{\cos \pi t}\,dt\right) + \frac{d}{dx}e^{\pi x} = -\frac{1}{2}\frac{d}{dx}\left(\int_{0}^{x^2-1} \frac{(t^2-1)(t+2)}{\cos \pi t}\,dt\right) + \pi e^{\pi x}$$
$$= -\frac{1}{2}\frac{[(x^2-1)^2-1][(x^2-1)+2]}{\cos[\pi(x^2-1)]}\cdot\frac{d}{dx}(x^2-1) + \pi e^{\pi x} = -\frac{1}{2}\frac{[(x^2-1)^2-1][(x^2-1)+2]}{\cos[\pi(x^2-1)]}\cdot 2x + \pi e^{\pi x}.$$

It's not necessary to simplify this, for we can simply plug in $x = -1$ to find

$$F'(-1) = -\frac{1}{2}\frac{[((-1)^2-1)^2-1][((-1)^2-1)+2]}{\cos[\pi((-1)^2-1)]}\cdot(-2) + \pi e^{-\pi} = -\frac{1}{2}\frac{(-1)(2)}{\cos(0)}(-2) + \frac{\pi}{e^{\pi}} = \frac{\pi}{e^{\pi}} - 2.$$

We now have all the necessary ingredients, and can plug these into the general form for the equation of a tangent line: $y - \frac{1}{e^{\pi}} = \left(\frac{\pi}{e^{\pi}} - 2\right)(x+1)$, or with the aid of a calculator, $y - 0.043 = -1.864(x+1)$.

24. Let $g(x) = x^2 - \int_0^{2x} \frac{t}{\sqrt{t^2+5}}\,dt$.
Find an equation for the tangent line to the curve $y = g(x)$ at $x = 1$. The equation of the tangent line will again have the form $y - y_0 = m(x - x_0)$ where $x_0 = 1$, $y_0 = g(x_0) = g(1)$ and $m = g'(x_0) = g'(1)$. We have to compute $g(1)$ and $g'(1)$; for the former, we compute the integral using the substitution $u = t^2 + 5$:

$$g(1) = 1 - \int_0^2 \frac{t}{\sqrt{t^2+5}}\,dt = 1 - \int_0^2 \frac{1}{\sqrt{u}}\left(\frac{1}{2}\frac{du}{dt}\right)dt = 1 - \frac{1}{2}\int_{u(0)}^{u(2)} u^{-1/2}\,du = 1 - \sqrt{u}\Big|_5^9 = 1 - (3 - \sqrt{5}) = \sqrt{5} - 2,$$

and for the latter we use part 1 of the fundamental theorem:

$$g'(x) = 2x - \frac{d}{dx}\int_0^{2x} \frac{t}{\sqrt{t^2+5}}\,dt = 2x - \left(\frac{2x}{\sqrt{(2x)^2+5}}\right)\cdot\frac{d}{dx}2x = 2x - \frac{4x}{\sqrt{4x^2+5}},$$

so $g'(1) = 2 - \frac{4}{\sqrt{4+5}} = 2/3$. Thus the equation of the tangent line is $y - (\sqrt{5} - 2) = \frac{2}{3}(x - 1)$.

25. Let

$$F(x) = \begin{cases} x + 3 & x \le -2, \\ 1 + \sqrt{1 - (x+1)^2} & -2 \le x \le 0, \\ 1 - \sqrt{x} & x \ge 0. \end{cases}$$

Find $\int_{-2}^{0} F(x)\,dx$, $\int_{-3}^{0} F(x)\,dx$ and $\int_{-3}^{1} F(x)\,dx$. In the interval $-2 \le x \le 0$ we have $F(x) = 1 + \sqrt{1-(x+1)^2}$, so the first integral can be rewritten as $\int_{-2}^{0} F(x)\,dx = \int_{-2}^{0}\left(1 + \sqrt{1-(x+1)^2}\right)dx$. This can be computed geometrically: notice that if $y = 1 + \sqrt{1-(x+1)^2}$, then $(x+1)^2 + (y-1)^2 = 1$, which is the equation of a circle of radius 1 centered at $(-1, 1)$. The interval $-2 \le x \le 0$ encompasses the whole circle, of which $F(x)$ represents the upper half, so the integral is equal to the area between the upper half of the circle and the x-axis: this is half the area of the circle, plus the area of a rectangle of height 1, width 2 (graph it, you'll see what I mean). So $\int_{-2}^{0} F(x)\,dx = \int_{-2}^{0}\left(1 + \sqrt{1-(x+1)^2}\right)dx = \frac{1}{2}\pi(1^2) + (1)(2) = \pi/2 + 2$.

To find $\int_{-3}^{0} F(x)\,dx$ we must consider separately the intervals in which $F(x)$ is defined by different equations:

$$\int_{-3}^{0} F(x)\,dx = \int_{-3}^{-2} F(x)\,dx + \int_{-2}^{0} F(x)\,dx = \int_{-3}^{-2}(x+3)\,dx + \int_{-2}^{0}\left(1 + \sqrt{1-(x+1)^2}\right)dx.$$

We just computed the second of these two integrals: it's $\pi/2 + 2$. And the first is easy: $\int_{-3}^{-2}(x+3)\,dx = \left(\frac{x^2}{2} + 3x\right)\Big|_{-3}^{-2} = 1/2$. You could also have done this geometrically, since the curve $y = x + 3$, for x

between -3 and -2, forms a right triangle with base and height each equal to 1. So $\int_{-3}^{0} F(x)\,dx = 1/2 + (\pi/2+2) = 5/2 + \pi/2$.

To find $\int_{-3}^{1} F(x)\,dx$ we must again split it into multiple intervals and use previous results:

$$\int_{-3}^{1} F(x)\,dx = \int_{-3}^{0} F(x)\,dx + \int_{0}^{1} F(x)\,dx.$$

The first integral is the one we just computed: it equals $5/2+\pi/2$. For the second, we have $F(x) = 1-\sqrt{x}$ when $0 \le x \le 1$, so $\int_0^1 F(x)\,dx = \int_0^1 (1-\sqrt{x})\,dx$. Geometrically this represents a parabola turned on its side, but we don't know how to find the area of a parabola, so we can't compute the integral geometrically. Fortunately, we do know how to do this using the fundamental theorem:

$$\int_0^1 (1-\sqrt{x})\,dx = \int_0^1 \left(1 - x^{1/2}\right)\,dx = \left(x - \frac{2}{3}x^{3/2}\right)\Big|_0^1 = \frac{1}{3}.$$

So $\int_{-3}^{1} F(x)\,dx = (5/2+\pi/2) + 1/3 = 17/6 + \pi/2$.

26. $\displaystyle\int_0^{\pi/4} \frac{(\cos^3 x - 1)^2}{\cos^2 x} \sin x\,dx$. The integrand is defined and continuous in the given interval since $\cos^2 x > 0$ when $0 \le x \le \pi/4$. Substitute $u = \cos x$, $\frac{du}{dx} = -\sin x$:

$$\begin{aligned}\int_0^{\pi/4} \frac{(\cos^3 x - 1)^2}{\cos^2 x} \sin x\,dx &= \int_0^{\pi/4} \frac{(u^3-1)^2}{u^2}\left(-\frac{du}{dx}\right)\,dx = -\int_0^{\pi/4} \frac{(u^3-1)^2}{u^2}\frac{du}{dx}\,dx \\ &= -\int_{\cos(0)}^{\cos(\pi/4)} \frac{(u^3-1)^2}{u^2}\,du = -\int_1^{\sqrt{2}/2} \frac{u^6 - 2u^3 + 1}{u^2}\,du \\ &= -\int_1^{\sqrt{2}/2} \left(u^4 - 2u + u^{-2}\right)\,du = -\left(\frac{u^5}{5} - u^2 - \frac{1}{u}\right)\Bigg|_1^{\sqrt{2}/2} \\ = -\left[\frac{1}{5}\left(\frac{2^{5/2}}{2^5}\right) - \frac{1}{2} - \sqrt{2}\right] - \left[-\left(\frac{1}{5} - 1 - 1\right)\right] &= \left[-\frac{1}{5}\left(\frac{\sqrt{32}}{32}\right) + \frac{1}{2} + \sqrt{2}\right] - \frac{9}{5} = -\frac{13}{10} + \frac{39\sqrt{2}}{40}.\end{aligned}$$

27. Prove that $\displaystyle\int_0^2 e^{\sqrt{x^2-2x+5}}\,dx \ge 2e^2$.

In a problem of this sort, you don't know how to compute the integral explicitly, but you can produce an estimate by finding the maximum/minimum values of the integrand. If you can find a number m such that $e^{\sqrt{x^2-2x+5}} \ge m$ for all x from 0 to 2, then it's clear that $\int_0^2 e^{\sqrt{x^2-2x+5}}\,dx \ge \int_0^2 m\,dx = mx\Big|_0^2 = 2m$. So the question is simply: what is m, i.e. the minimum value of $e^{\sqrt{x^2-2x+5}}$ when $0 \le x \le 2$? This becomes a much simpler question if you notice that the function e^u is always increasing with u, so $e^{\sqrt{x^2-2x+5}}$ should attain its minimum at the same value for x that $\sqrt{x^2-2x+5}$ does. And we can take the same argument one step further: $\sqrt{x^2-2x+5}$ attains its minimum at the same x value as x^2-2x+5. Let's call $g(x) = x^2-2x+5$, and find its minimum in the interval $0 \le x \le 2$. $g'(x) = 2x-2$, which is 0 when $x = 1$, thus 1 is a critical point, and we can easily check by plugging in x values on either side of 1 that it's a local minimum. It is, in fact, the *only* local minimum, since the equation $g'(x) = 2x-2 = 0$ has only one solution. We conclude therefore that the minimum value of x^2-2x+5 in our interval is $g(1) = 4$. It follows immediately that the minimum value of $\sqrt{x^2-2x+5}$ is $\sqrt{4} = 2$, and the minimum value of $e^{\sqrt{x^2-2x+5}}$ is thus e^2. So for all x with $0 \le x \le 2$, $e^{\sqrt{x^2-2x+5}} \ge e^2$, and thus

$$\int_0^2 e^{\sqrt{x^2-2x+5}}\,dx \ge \int_0^2 e^2\,dx = 2e^2.$$

28. $\dfrac{d}{dx}\sqrt{\ln(1-x^2)}$. You could try to do this using the chain rule, and the answer you'd get would be $-\frac{x}{(1-x^2)\sqrt{\ln(1-x^2)}}$. But you'd be wrong, for the following reason: $(1-x^2)$ is always less than or equal

to 1, thus $\ln\left(1-x^2\right) \leq 0$ for all x. But $\sqrt{\ln(1-x^2)}$ is only defined when $\ln\left(1-x^2\right) \geq 0$, which is only true when $x = 0$. So the function is undefined at every point except $x = 0$, and you cannot take the derivative a function whose domain is only one point. The correct answer is **undefined**.

29. Let

$$f(x) = \begin{cases} (x-2)^2 & x \leq 2 \\ \sqrt{x-2} & x \geq 2 \end{cases}$$

Find $f'(0)$, $f'(6)$ and $\int_0^4 f(x)\,dx$. At $x = 0$, $x \leq 2$, thus $f(x) = (x-2)^2$, $f'(x) = 2(x-2)$ and $f'(0) = -4$. Similarly at $x = 6$, $x \geq 2$, so $f(x) = \sqrt{x-2}$ and $f'(6) = \frac{1}{2\sqrt{x-2}} = \frac{1}{4}$. (A question to think about: what if we asked you for $f'(2)$?)

To find $\int_0^4 f(x)\,dx$ we must split it into two intervals:

$$\begin{aligned} \int_0^4 f(x)\,dx &= \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx \\ &= \int_0^2 (x-2)^2\,dx + \int_2^4 \sqrt{x-2}\,dx = \left.\frac{(x-2)^3}{3}\right|_0^2 + \left.\frac{2}{3}(x-2)^{3/2}\right|_2^4 = \frac{8}{3} + \frac{4\sqrt{2}}{3}. \end{aligned}$$

30. $\left|\int_0^4 (x^2-5x+6)\,dx\right| = \left|\left(\frac{x^3}{3} - \frac{5}{2}x^2 + 6x\right)\Big|_0^4\right| = \left|\frac{16}{3}\right| = \frac{16}{3}$.

31. $\int_0^4 |x^2-5x+6|\,dx$. We must first find out when x^2-5x+6 is positive and when it's negative, then we can define the function $|x^2-5x+6|$ "piecewise" as in problem 29. By solving the quadratic equation $g(x) = x^2-5x+6 = (x-2)(x-3) = 0$ we find that $g(x) = 0$ when x is 2 or 3; then we can plug in different values of x to see that $g(x) > 0$ when $x < 2$ or $x > 3$, and $g(x) < 0$ when $2 < x < 3$. When $g(x) < 0$, $|g(x)| = -g(x)$, otherwise $|g(x)| = g(x)$, so we can write:

$$f(x) = |g(x)| = |x^2-5x+6| = \begin{cases} x^2-5x+6 & \text{if } x \leq 2 \text{ or } x \geq 3, \\ -(x^2-5x+6) & \text{if } 2 \leq x \leq 3. \end{cases}$$

Thus,

$$\begin{aligned} &\int_0^4 |x^2-5x+6|\,dx = \int_0^2 \left(x^2-5x+6\right)\,dx + \int_2^3 -\left(x^2-5x+6\right)\,dx + \int_3^4 \left(x^2-5x+6\right)\,dx \\ &= \left(\frac{x^3}{3} - \frac{5}{2}x^2 + 6x\right)\Big|_0^2 - \left(\frac{x^3}{3} - \frac{5}{2}x^2 + 6x\right)\Big|_2^3 + \left(\frac{x^3}{3} - \frac{5}{2}x^2 + 6x\right)\Big|_3^4 = \frac{14}{3} - \left(-\frac{1}{6}\right) + \frac{5}{6} = \frac{17}{3}. \end{aligned}$$

Notice that this is greater than the answer to problem 30, as one would expect.

32. $\int_{-2}^2 |x^2-1|\,dx$. As in problem 31 we must find out when x^2-1 is positive or negative, and we get:

$$|x^2-1| = \begin{cases} x^2-1 & \text{if } x \leq -1 \text{ or } x \geq 1, \\ -(x^2-1) & \text{if } -1 \leq x \leq 1. \end{cases}$$

Thus

$$\begin{aligned} \int_{-2}^2 |x^2-1|\,dx &= \int_{-2}^{-1} (x^2-1)\,dx + \int_{-1}^1 -(x^2-1)\,dx + \int_1^2 (x^2-1)\,dx \\ &= \left(\frac{x^3}{3} - x\right)\Big|_{-2}^{-1} - \left(\frac{x^3}{3} - x\right)\Big|_{-1}^{1} + \left(\frac{x^3}{3} - x\right)\Big|_1^2 = \frac{4}{3} - \left(-\frac{4}{3}\right) + \frac{4}{3} = 4. \end{aligned}$$