Calculus II
Spring 2004
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Extra Credit Assignment 2, due Thursday May 6

The following is worth the equivalent of an extra 50-point homework assignment. You may deliver the solution to the bin outside my office, at 603 Warren Weaver Hall. Be aware also that something related to this is likely to turn up as a bonus question on the final.

In the last lecture we discussed the differential equation for a frictionless spring (or pendulum) that is being pushed back and forth at a fixed angular frequency α—this is called an *undamped driven harmonic oscillator.* Here's the same equation with friction included (the *damped driven harmonic oscillator*):

$$\theta''(t) = -k\theta(t) - b\theta'(t) + F\cos(\alpha t) \tag{1}$$

Here k, b and F are all positive constants, and α is also a constant. The left hand side of the equation is acceleration (which Newton says is proportional to force), and each term on the right hand side has an interpretation as one of the three forces acting on the spring:

- $-k\theta$ is the *restoring force*, which pushes the spring back toward its equilibrium position
- $-b\theta'$ is the *friction*, a force proportional to the speed and in the opposite direction
- $F\cos(\alpha t)$ is the *driving force*, caused, e.g. by my hand pushing the spring back and forth at a fixed rate

As in the undamped case, we expect there should be a solution which oscillates at the same rate as the driving force, but we can't quite get away with assuming the solution is as simple as $A\cos(\alpha t)$. We'll need to allow for a *phase shift*:

$$\theta(t) = A\cos(\alpha t - \phi)$$

The extra ϕ causes $\theta(t)$ to lag behind $F\cos(\alpha t)$, i.e. they have the same frequency, but reach their maxima and minima at different times. Your task is to find out what the constants A and ϕ have to be in order for $\theta(t)$ to satisfy Equation 1. You can express A and ϕ in terms of the constants in the equation:

- the driving force strength F
- the driving frequency α
- the friction strength b
- the restoring force strength k—or more conveniently, the natural vibrational frequency $\omega_0 = \sqrt{k}$

In principle this is straightforward, but you'll find it extremely difficult without using complex numbers. Here's my advice: look for a *complex-valued* solution of the equation

$$\theta''(t) = -k\theta(t) - b\theta'(t) + Fe^{i\alpha t} \tag{2}$$

Guess that it should take the form $\theta(t) = Ae^{i(\alpha t - \phi)}$, then find A and ϕ. This works because if you take the real part of both sides of Equation 2, assuming $\theta(t)$ has this complex form, what you get is precisely Equation 1 with $\theta(t)$ in the real form described above.
Now that you've figured that out, what choice of α gives you the largest vibration for $\theta(t)$? (You should already know the answer to this on intuitive grounds, so this way you can make sure your results look reasonable.)