Homework 6

1) $1 = z^5 = r^5 (\cos(5\theta) + i \sin(5\theta))$

Hence $|1| = |r^5 (\cos(5\theta) + i\sin(5\theta))|$

$= |r|^5 \, |\cos(5\theta) + i \sin(5\theta)|$

$= r^5$

Therefore $r = 1$. Further θ has to satisfy

$$\cos(5\theta) + i \sin(5\theta) = 1$$

implying $\theta = j \cdot \frac{2\pi}{5}$ for $j = 0, 1, 2, 3, 4$

$$z_j = \cos\left(2\pi \frac{j}{5}\right) + i \sin\left(2\pi \frac{j}{5}\right)$$

$j = 0, \ldots 4$ are the solutions

b) $0 = z^2 + z + 1 = z^2 + z + \frac{1}{4} + \frac{3}{4}$

$= (z + \frac{1}{2})^2 + \frac{3}{4}$.

Hence $(z + \frac{1}{2})^2 = -\frac{3}{4} = \left(\sqrt{\frac{3}{4}}\, i\right)^2$

implying $z + \frac{1}{2} = \pm \sqrt{\frac{3}{4}}\, i$

$z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

2)

$$\frac{5+3i}{6-i} = \frac{(5+3i)(6+i)}{37} = \frac{(30-3)+i(18+5)}{37}$$

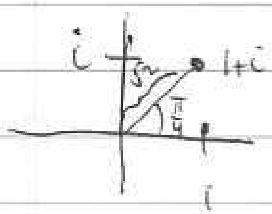
$$= \frac{27}{37} + \frac{23}{37} i$$

$$\frac{\pi+i}{5-i} = \frac{(\pi+i)(5+i)}{26} = \frac{(5\pi-1)+(5+\pi)i}{26}$$

$$= \frac{5\pi-1}{26} + \left(\frac{5+\pi}{26}\right) i$$

$$(1+i)^8 = \left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)\right)^8$$

$$= (\sqrt{2})^8\left(\cos\left(8\cdot\frac{\pi}{4}\right)+i\sin\left(8\cdot\frac{\pi}{4}\right)\right)$$



$$= 16\,(1 + i\cdot 0)$$

$$= 16$$

3) If a point w satisfies

$\Re(w) + \Im(w) = 0$ it has the form

$$w = x - ix$$

for a real number x. In polar coordinates

$$w = r\left(\cos\left(\tfrac{3}{4}\pi\right) + i\sin\left(\tfrac{3}{4}\pi\right)\right)$$

with $r \in \mathbb{R}$

If $w = z^2$ we must have

$$z = s\left(\cos\theta + i\sin(\theta)\right),$$

satisfying since

$$z^2 = s^2\left(\cos(2\theta) + i\sin(2\theta)\right),$$

that

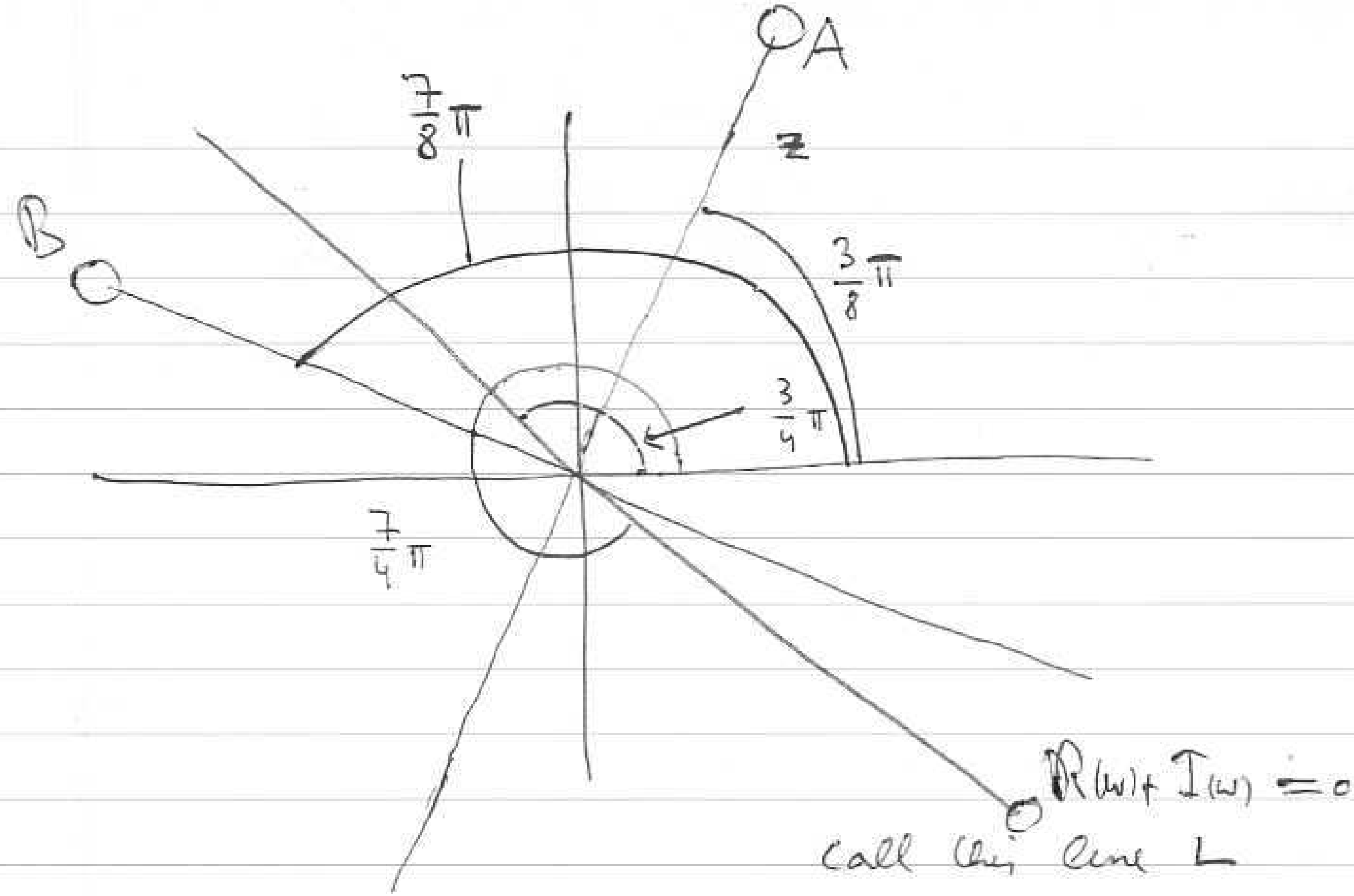
$$2\theta = \frac{3}{4}\pi$$

or $$2\theta = \frac{3}{4}\pi + \pi.$$

Hence $$\theta = \frac{3}{8}\pi$$ or

$$\theta = \frac{7}{8}\pi$$

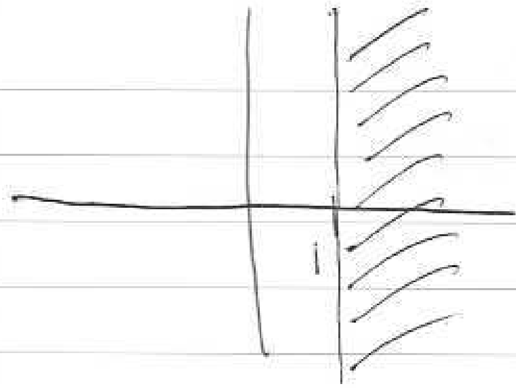
$$\{z \in \mathbb{C} \mid \Re(z^2) + \Im(z^2) = 0\} = \left\{ s(\cos\theta + i\sin\theta) \,\middle|\, s \in \mathbb{R},\ \theta = \tfrac{3}{8}\pi \text{ or } \tfrac{7}{8}\pi \right\}$$



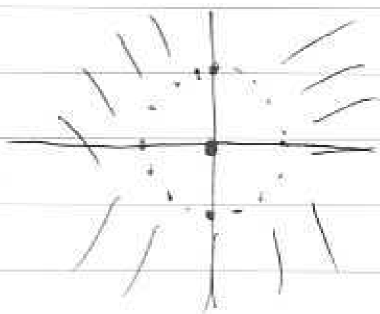
The lines A and B contain points z which under squaring end up in the Line L

Illustration of the proof of 3)

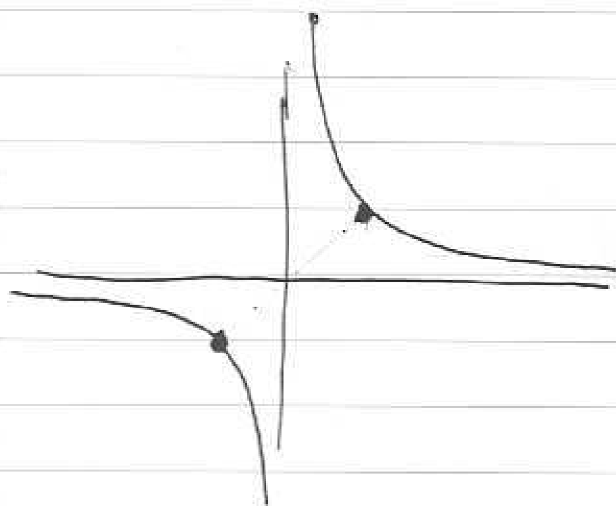
4)



$\Re(z) \geq 1$



$|z| > 1$



$\Re(z) \cdot \Im(z) = 1$

5) $\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$

Hence

$$\sin(-\theta) = -\theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \ldots$$

$$= -1\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right)$$

$$= (-1)\cdot\sin(\theta)$$

$$= -\sin(\theta)$$

6)

a) $\left(\dfrac{1}{1+\dfrac{1}{1+i}}\right)^2 = \left(\dfrac{1+i}{1+i+1}\right)^2$

$= \left(\dfrac{1+i}{2+i}\right)^2 = \dfrac{1+2i-1}{4-1+4i} = \dfrac{2i}{3+4i}$

$= \dfrac{2i\,(3-4i)}{9+16} = \dfrac{8+6i}{25}$

$= \dfrac{8}{25} + \dfrac{6}{25}\,i$

b) $|4+3i| = \sqrt{16+9} = 5$